Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 18
Generalization/Numerical Solution of Euler Lagrange Equations Part 6

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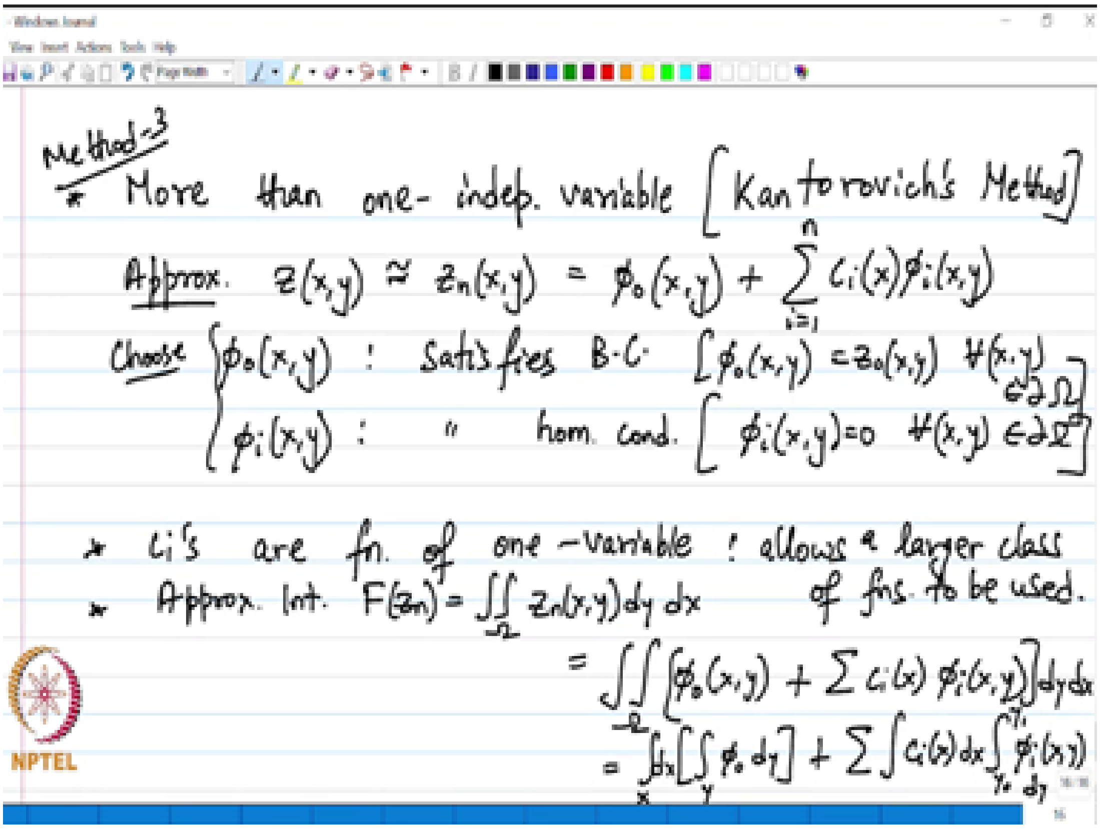

let me now go to another new method. Suppose we have also seen functionals which has more than one independent variable, in that case, the Ritz method changes to a much more general method known as the Kantorovich's method. So, let me call this as method 3, if we have more than one independent variable, we are going to use the so called Kantorovich's method.

What exactly is Kantorovich's method? So, it is an extension of the Ritz method, For example, we have a function of 2 variables, so we approximate our function of 2 variables as follows, $Z(x,y) \approx Z_n(x,y) = \phi_o(x,y) + \sum_{i=1}^{n} C_i(x)\phi_i(x,y)$ where my basis functions are both functions of x and y.

Again we choose $\phi_o(x,y)$ such that it satisfies the boundary condition, the same principle as we did with Ritz method or let me say that $\phi_o(x,y) = Z_o(x,y)\ \forall\ (x,y) \in \partial\Omega$.

Also I choose my $\phi_i(x,y)$ such that it satisfies homogenous condition $\phi_i(x,y) = 0\ \forall\ (x,y) \in \partial\Omega$ So, it vanishes on the boundary, since we have chosen C to be functions of one variable, it provides us to choose a larger class of functions. So, this method is the generalisation of the Ritz method has provided us with more flexibility here.

So, so as I said the major difference is C_i's are now function of one variable, they are now functions of one variable and it allows, it allows a larger, a larger class of functions to be used. Now, so so far we have seen the approximation. This question is, how are we you going to use this approximation? So, notice notice that my, my original integral is the double integral. So, I am going to first, I am going to first integrate one of the integrals. So, I am going to integrate, I am going to integrate let us say the inner integral or the y integral. let me say my approximate integral $F(Z_n) = \iint_{\Omega} Z_n(x,y)dydx$. So, we can easily approximate the y integral and replace z_n with its approximation. We see that this is going to give us the following

$$F(Z_n) = \iint_{\Omega} Z_n(x,y)dydx = \iint_{\Omega} \left[\phi_o(x,y) + \sum C_i(x)\phi_i(x,y)\right] dydx$$

$$= \int_x dx \left[\int_y \phi_o dy\right] + \sum \int C_i(x)dx \int_{y_o}^{y_1} \phi_i(x,y)dy$$

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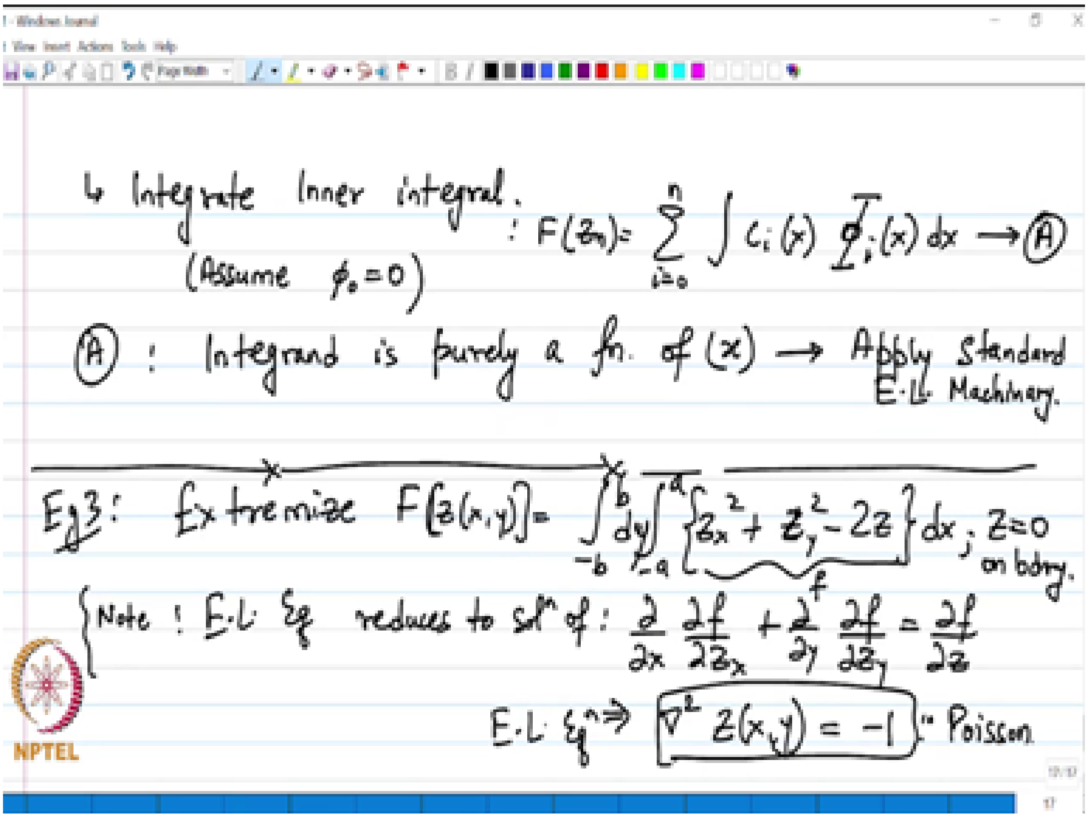

we integrate the inner integral, let us for the time being assume that that $\phi_o = 0$. Anyway it vanishes on the boundary. So, let us assume that ϕ the first family of the basis function is 0. So, notice that now my approximate integral becomes $F(Z_n) = \sum_{i=0}^{n} \int C_i(x)\Phi_i(x)dx$ **A**

Notice that the integrant in **A** is purely a function of x, now we have changed a functional of two variables to a functional of one variable x. So, from here we can start applying the standard Euler Lagrange Machinery.

To get the exact or solution with respect to the variable x or approximate solution with respect to the variables x y. So, let me just highlight through an example as to what I just said

Example 3 : Extremize $F(Z(x,y)) = \int_{-b}^{b} dy \int_{-a}^{a} \left\{Z_x^2 + Z_y^2 - 2Z\right\} dx$; Z = 0 on boundary

We know that the Euler Lagrange solution to integrands having two independent variables x and y is as follows

Note that my Euler Lagrange equation reduces to the solution of the following equation

$\frac{\partial}{\partial x}\frac{\partial f}{\partial Z_x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial Z_y} = \frac{\partial f}{\partial Z}$, assume that Z is 0 on the boundary, It is 0 on the boundary of this functional.

So, when we plug in the value of f which is this quantity, we see that this is also equal to $\bigtriangledown^2 z(x, y) = -1$ So, my Euler Lagrange Equation reduces to the Poison equation and all we need to do to find the extremal is to solve this Poison equation here. So, now we are now going to solve via the Kantorovich's method. Well, students can can definitely solve this Poison equation subject to the 0 boundary condition. We are now going to highlight the solution via the Kantorovich's method.

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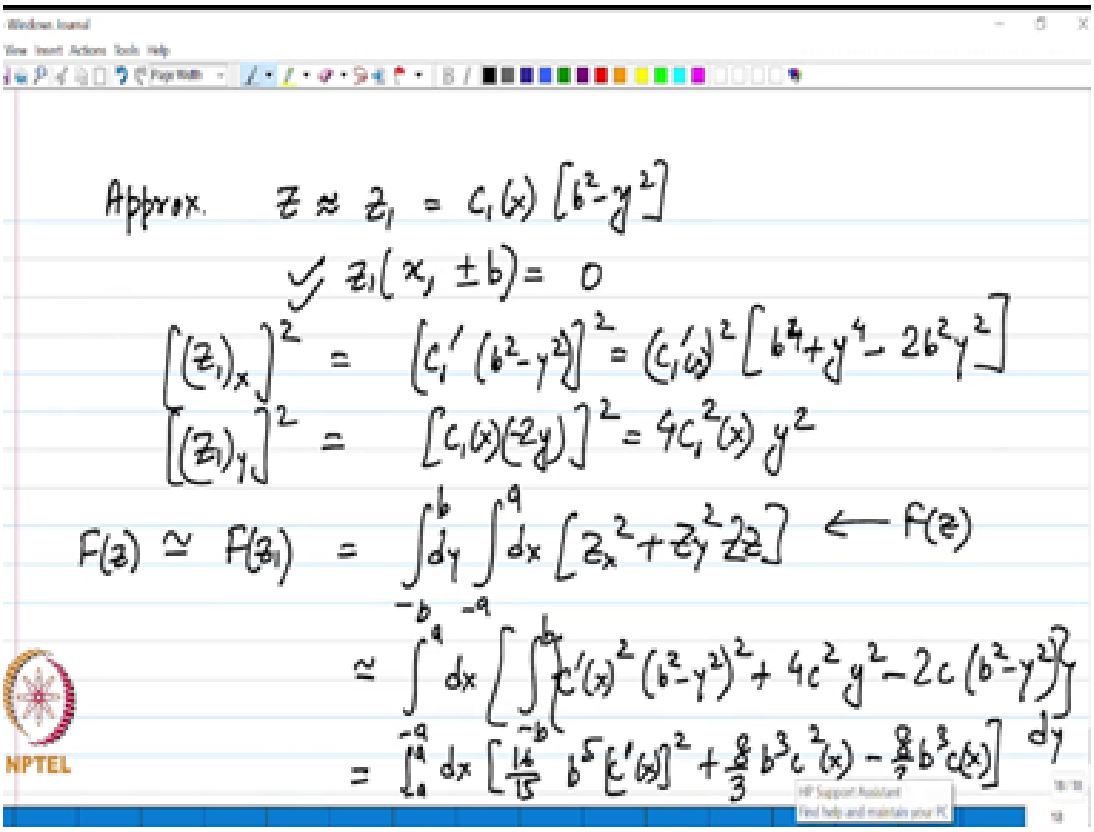

let me just approximate $z \approx Z_1 = C_1(x)[b^2 - y^2]$ We note that $Z_1(x, \pm b) = 0$, now our approximation is such that we are now left with two more boundary condition that is at $x \pm a$ which we will utilise later.

So, we have already utilised two boundary conditions

$$\Rightarrow [(Z_1)_x]^2 = \left[C_1^{'}(b^2 - y^2)\right]^2 = (C_1^{'}(x))^2\left[b^4 + y^4 - 2b^2y^2\right]$$

$$[(Z_1)_y]^2 = [C_1(-2y)]^2 = 4C_1^2(x)y^2$$

$$F(Z) \approx F(Z_1) = \int_{-b}^{b} dy \int_{-a}^{a} dx\left[Z_x^2 + Z_y^2 2Z\right]$$

$$\approx \int_{-a}^{a} dx\left[\int_{-b}^{b}\left\{C^{'}(x)^2(b^2 - y^2)^2 + 4C^2y^2 - 2C(b^2 - y^2)\right\}\right]$$

$$= \int_{-a}^{a} dx\left[\frac{16}{15}b^5\left[C^{'}(x)\right]^2 + \frac{8}{3}b^3C^2(x) - \frac{8}{3}b^3C(x)\right]$$

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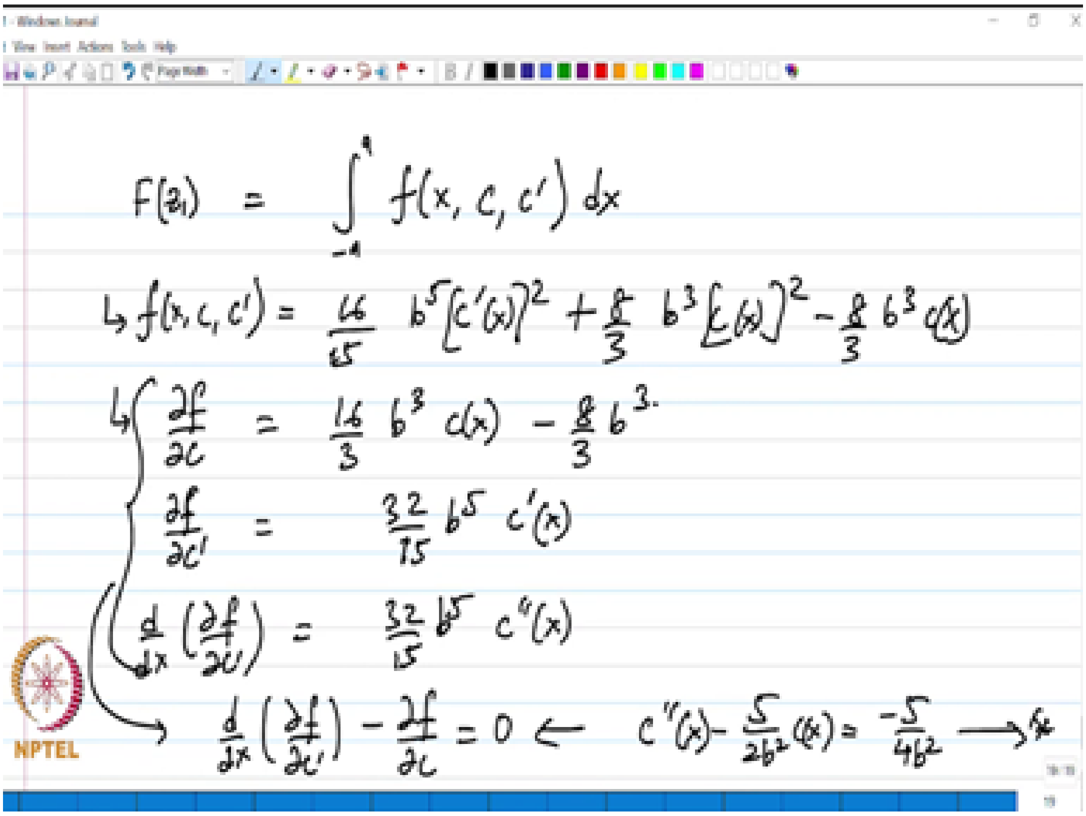

$$F(z_1) = \int_{-a}^{a} f(x, C, C')dx$$

$$f(x, C, C') = \frac{16}{15}b^5\left[C'(x)\right]^2 + \frac{8}{3}b^3C^2(x) - \frac{8}{3}b^3C(x)$$

$$\frac{\partial f}{\partial C} = \frac{16}{3}b^3C(x) - \frac{8}{3}b^3$$

$$\frac{\partial f}{\partial C'} = \frac{32}{15}b^5C'(x)$$

Let me further differentiate with respect to x so that is what we get. So, then well let me keep these two term separate because I need both the terms.

$$\frac{d}{dx}\frac{\partial f}{\partial C'} = \frac{32}{15}b^5C''(x)$$

We use Euler Lagrange Equation $\frac{d}{dx}\frac{\partial f}{\partial C'} - \frac{\partial f}{\partial C} = 0 \Rightarrow \quad C''(x) - \frac{5}{2b^2}C(x) = -\frac{5}{4b^2}$ $\quad *$

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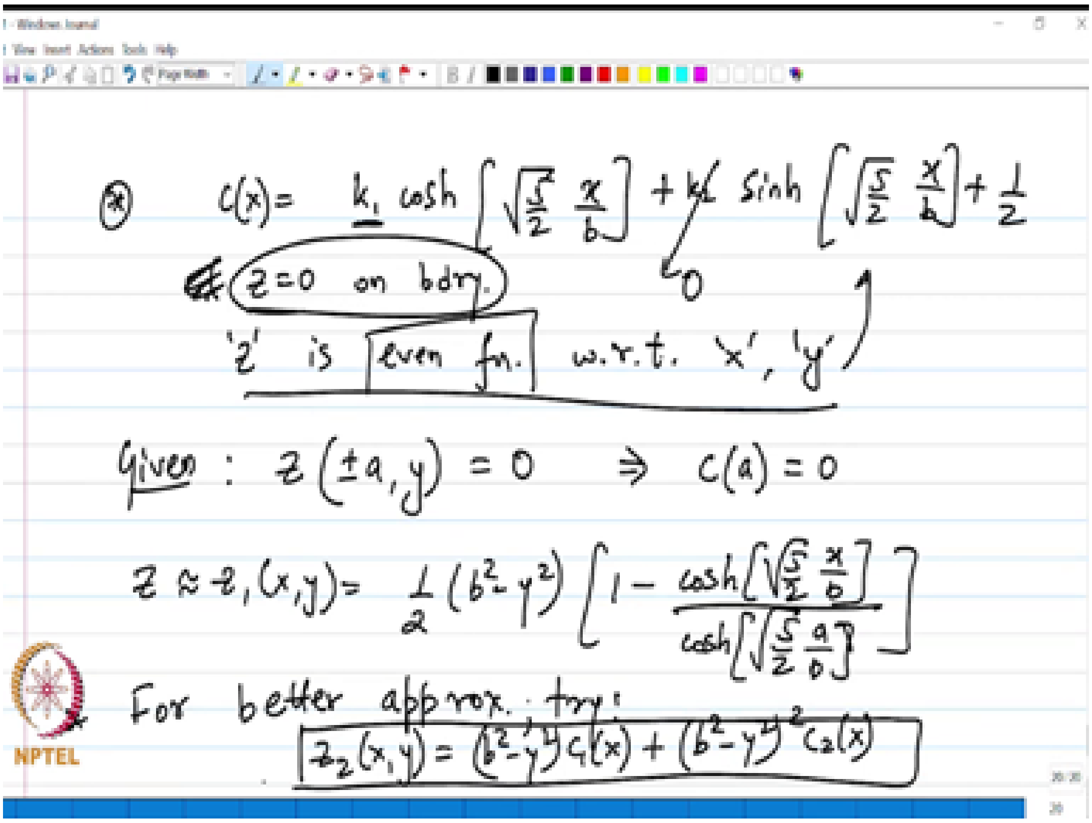

Now, students can notice that this is a constant coefficient non-homogeneous ODE and the solution to the * is as follows

$$C(x) = k_1 \cosh\left[\sqrt{\frac{5}{2}}\frac{x}{b}\right] + k_2 \sinh\left[\sqrt{\frac{5}{2}}\frac{x}{b}\right] + \frac{1}{2}$$

Z is 0 on the boundary, so z is an even function even function with respect to both x and y due to the presence of the boundary condition that we have and also the type of the functional that we have. That is easy to check and so which means that we can completely using this condition we can completely eliminate k_2. Even if you do not know the symmetry, even if you do not know that z is an even function, we can use both the boundary conditions, not a problem. Right now I am just simplifying.

Now we only have one parameter to solve for that is k_1 and we also know that we also know that this is given that $z(\pm a, y) = 0 \;\Rightarrow\; C(a) = 0$, the solution vanishes on the boundary. So, I can very well use the fact that c(a) = 0 is 0, do not need to use the other boundary condition because we have already used the fact that z is an even function. Once we do that, we see that the solution is

$$z \approx z_1(x,y) = \frac{1}{2}(b^2 - y^2)\left[1 - \frac{\cosh\sqrt{\frac{5}{2}}\frac{x}{b}}{\cosh\sqrt{\frac{5}{2}}\frac{a}{b}}\right]$$

Notice that this vanishes on the boundary x = a. Now, I end my discussion here by mentioning that we could possibly use more and more approximation for a better better resolution.

For better approximation, we could possibly try this form of the approximate function $Z_2(x,y) = (b^2 - y^2)C_1(x) + (b^2 - y^2)^2 C_2(x)$.

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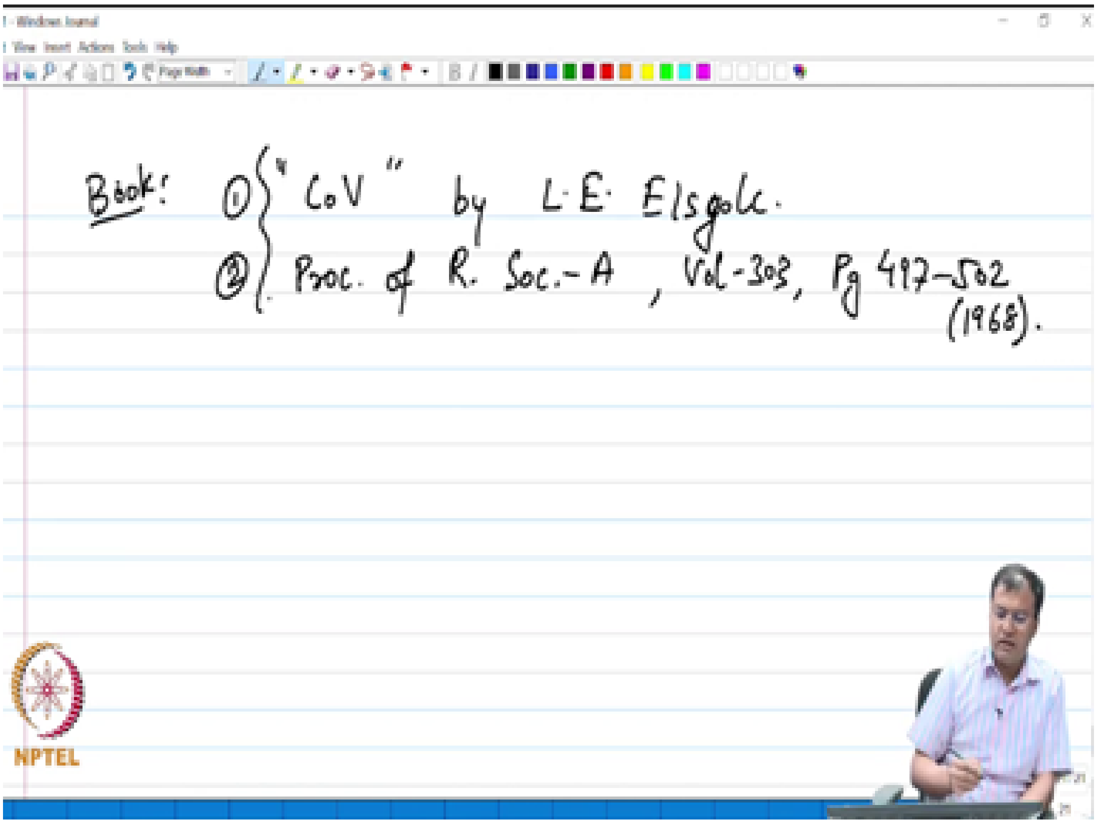

I end my discussion by giving a reference in which a more and more numerical methods and better numerical methods are prescribed. The students are asked to refer to the book Calculus of variations by L E Elsgolc and then there is another paper in Proceedings of Royal Society A volume 303 page 497 to 502 and this is a very classic paper in 1968. This mentions some of the numerical methods to the Euler Lagrange Equation and in the next lecture, I am going to talk about isoperimetric problem or constrained optimization.