Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 17

Generalization/Numerical Solution of Euler Lagrange Equations Part 5

We are going to end the discussion on the development of the theory in this case by giving few more tips.

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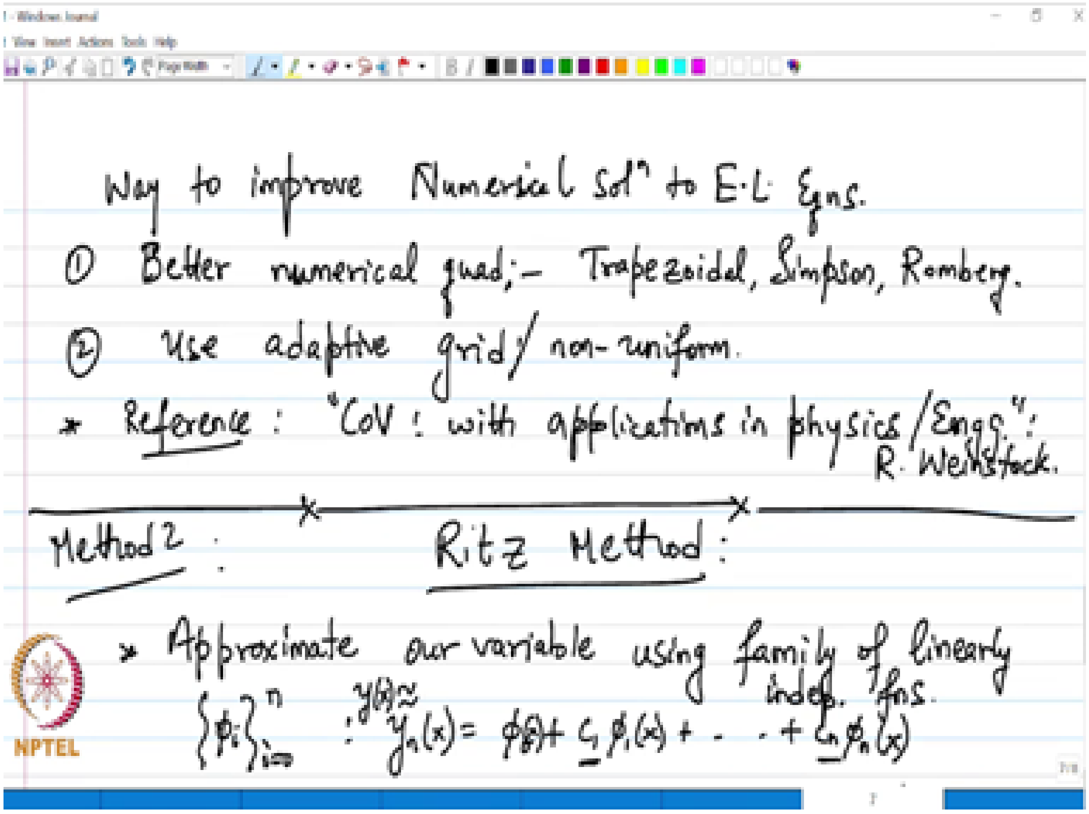

let me just highlight few more tips. So, ways the improve the Euler or Numerical solution, numerical solution to Euler Lagrange method, some of the natural tips would be to increase or make the grid finer or to increase the number of grid points or so, well so that is the standard suggestion but some of the non-standard suggestions are to use better quadrature rules.

Here we have just used rectangle rule to change the integral into summation. So, we could use better numerical quadrature. Some of the quadratures available are trapezoidal, we could use trapezoidal quadrature or we could use Simpson's rule or we could use even higher order quadrature like the Romberg rule or we could use adaptive grids.

So far the examples I have shown is for a uniform grid. We could also use adaptive grids. Grids will become finer where the functions changes rapidly and non-uniform or adaptive grids. So, more details can be found in this text books. So, students are asked to refer to this text book for more details on the numerical solutions of Euler Lagrange via the Euler's and other higher order methods.

This book is has the title Calculus of Variation with Applications in Physics and Engineering by Robert Weinstock. So, what we have is the following. let us look at another method or the numerical solution.

The method that I intend to highlight is the Ritz method, Now, what exactly is this method?

This method tells us that we could rather than approximating by taking the values of the function at finite number of grid points. We could use set of basis function to approximate our extremum function. So, we could use a set of orthogonal basis function and expand our functional in terms of the basis function. So, the idea is as follows.

We are going to approximate our variable using family of linearly independent functions and let us say our family is as follows: $\{\phi_i\}_{i=0}^{n}$, where our function is $y_n(x)$.

So, I am going to approximate y, so y the unknown of the problem is going to be approximated by $y_n(x) = C_1\phi_1(x) + ... + C_n\phi_n(x)$ so then again this sort of an approximation again reduces to the problem of function optimization where the unknowns now are these constants C_1 to C_n.

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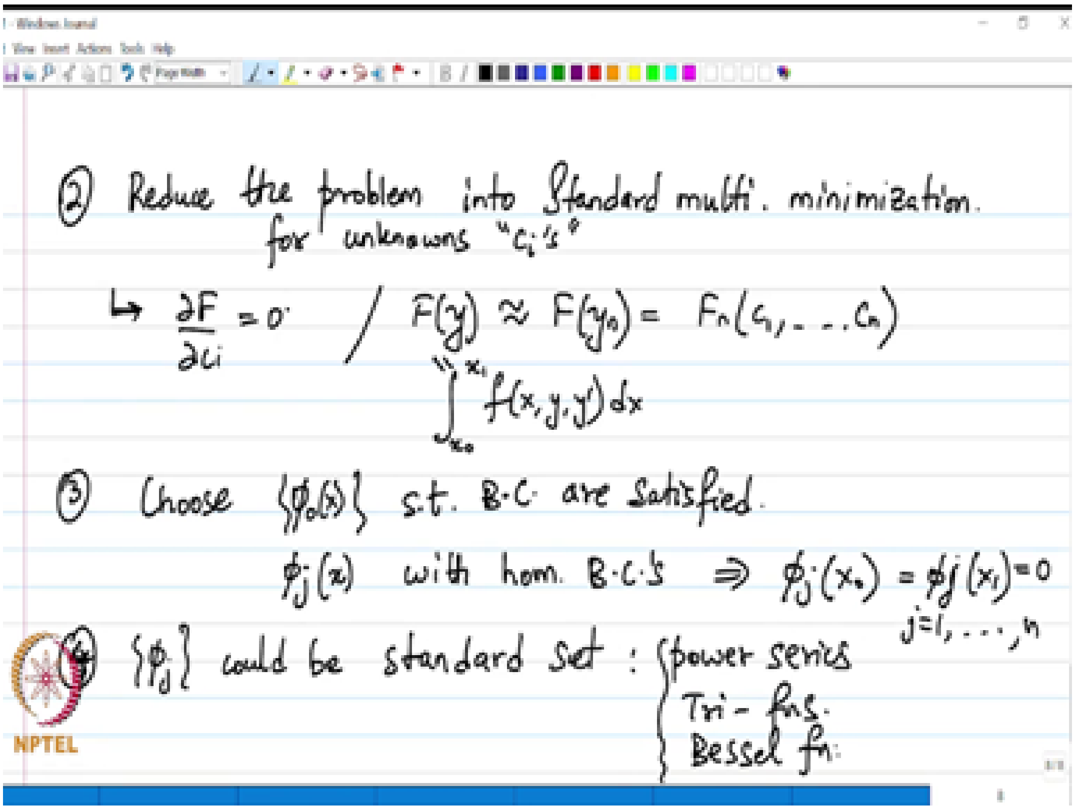

We reduce the problem into standard multivariate minimization problem for unknowns $C_i's$. Solution and all we do is differentiate our functional with respect to the $C_i's$ and set it equal to 0.

So, where my function is now an approximate function of y_n. so y has been approximated by y_n, which means this particular quantity to begin with was an integral. So, this was a functional of this integral of this following quantity. And this is now a function of the variable C_1 to C_n. So, then, well, let me just write down all the points and I am going to highlight this method with an example.

So, further we choose in our choice of ϕ_i, notice that, I have started with ϕ_o and I has started with a coefficient 1 and ϕ_1 to ϕ_n. So, we choose $\phi_o(x)$ such that the boundary conditions are satisfied and further we choose $\phi_j(x)$ such that with homogenous boundary condition.
Which means that we choose our ϕ_j such that $\phi_j(x_1) = 0$ where j = 1.....n, and then note that ϕ_j's could be from the standard set of orthogonal functions like power series or polynomials or it could be the trick

functions like sin and cos. It could also be the Bessel function and so on and so forth.

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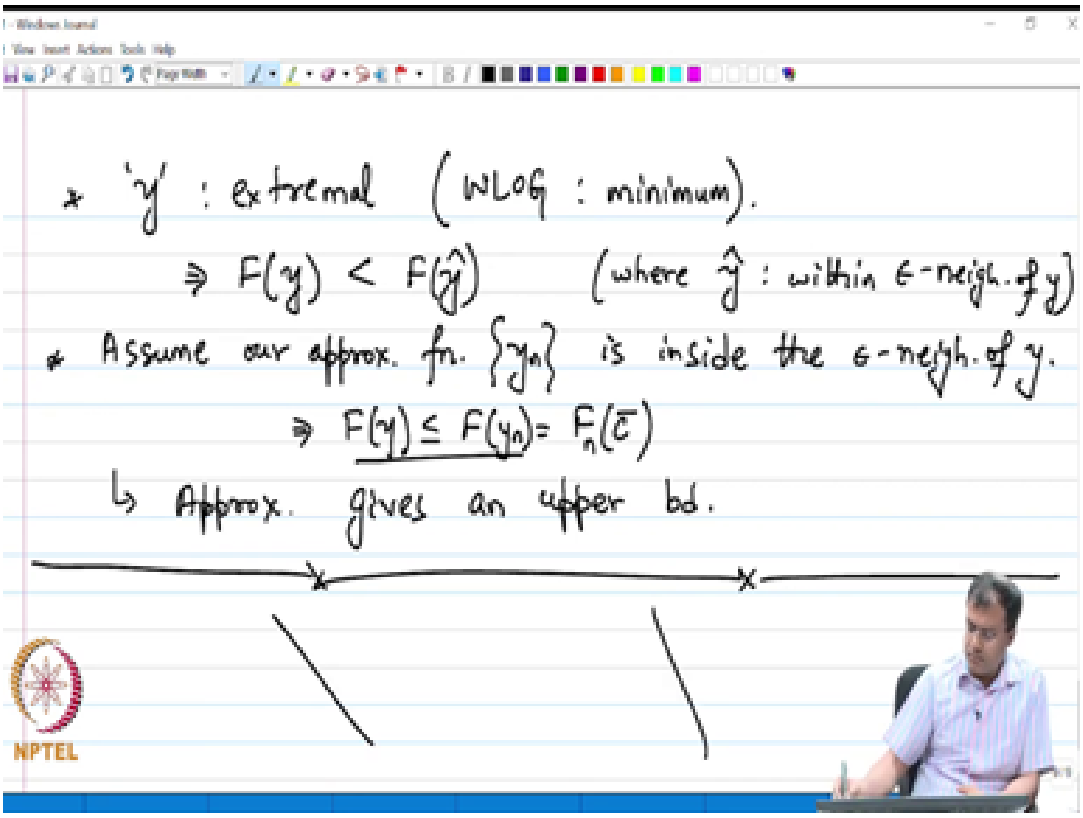

So, then we assume a little bit of analysis before we look at some examples, let us say that why is the extremal to the problem at hand, we assume that without loss of generality the extremal we are talking is minimum. So, in that case, we must have that the value of the functional $F(y) < f(\hat{y})$ where $\hat{y}$ is the perturbation of y where $\hat{y}$ is within ϵ neighbourhood of y, close enough to y.

What I want to show here is that the value of the functional which is the value of the function that we get from the Ritz method will be if it is very close to the original extremal found from the Euler Lagrange. Then the value of the function from the Ritz method will give an upper bond to the value of the functional or the exact value of the functional.

So, what I just said is the following: so, what I have is F(y), so we assume that our approximate solution, our approximate function y_n is inside the ϵ neighbourhood of y, which means the approximation from the Ritz method is sufficiently close enough to the exact extremal. So, in that case, what we have is $F(y) \leq F(y_n)$ which is the value obtained by evaluating the values of these $C_i' s$

As I just said the approximation gives an upper bound as we can see from here, so care must be taken for a careful choice of $\phi' s$ and not only that the higher the value of $\phi' s$, we expect that the closer is the approximate solution to the exact solution. So, let us let us look at an example.

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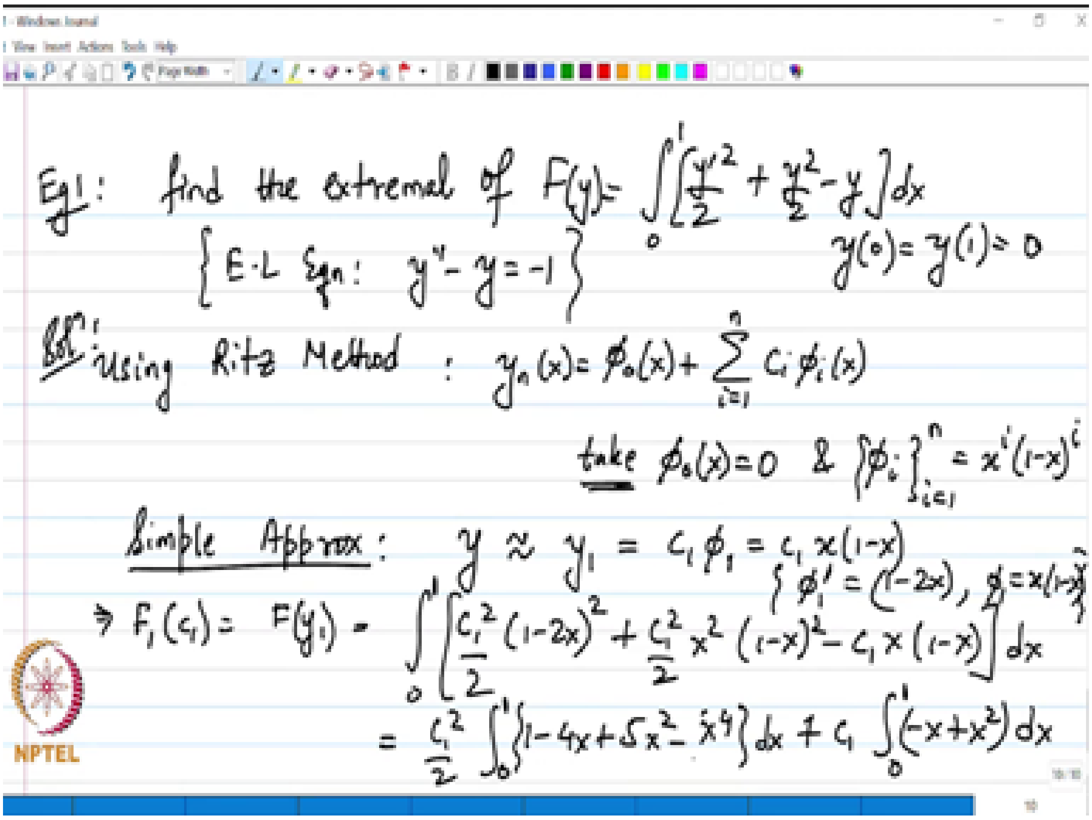


Eg1: find the extremal of $F(y)=\int_0^1\left[\frac{y'^2}{2}+\frac{y^2}{2}-y\right]dx$, $y(0)=y(1)=0$

$\{$E-L Eqn: $y''-y=-1\}$

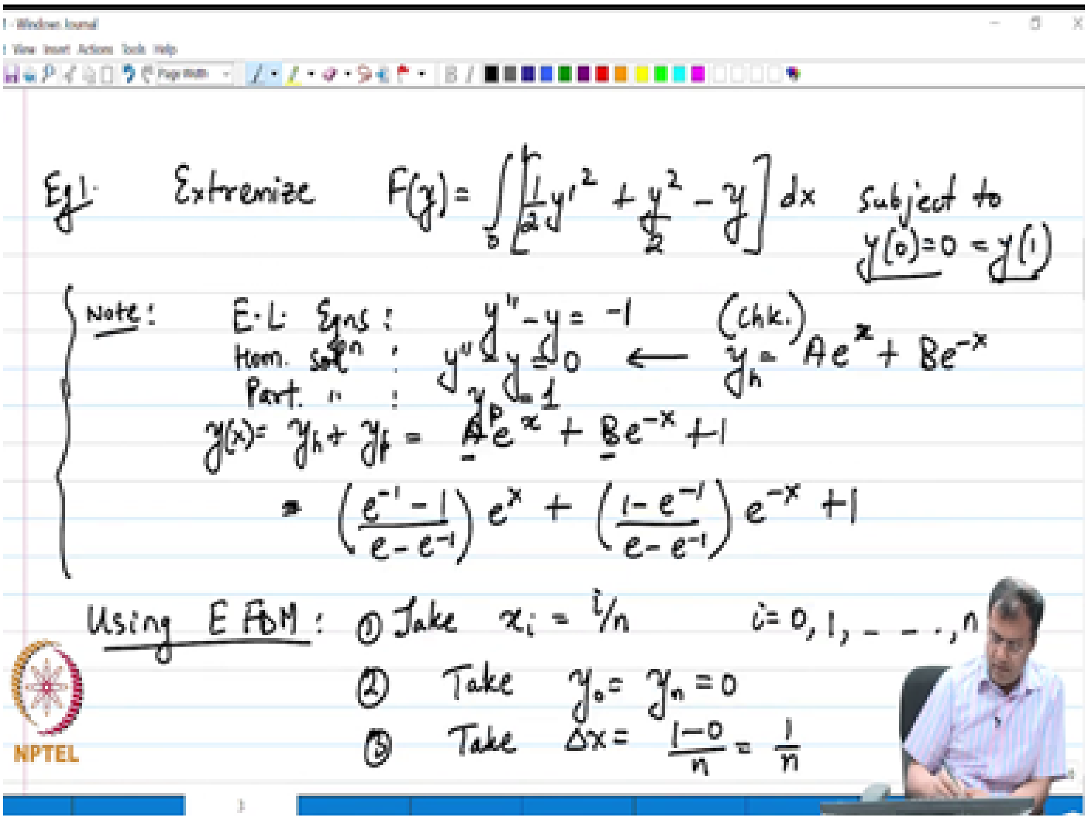
Sol: using Ritz Method: $y_n(x)=\phi_0(x)+\sum_{i=1}^{n}c_i\phi_i(x)$

take $\phi_0(x)=0$ & $\{\phi_i\}_{i=1}^{n}=x^i(1-x)^i$

Simple Approx: $y\approx y_1=c_1\phi_1=c_1x(1-x)$ $\{\phi_1'=(1-2x),\ \phi_1=x(1-x)\}$

$$\Rightarrow F_1(c_1)=F(y_1)=\int_0^1\left[\frac{c_1^2}{2}(1-2x)^2+\frac{c_1^2}{2}x^2(1-x)^2-c_1x(1-x)\right]dx$$

$$=\frac{c_1^2}{2}\int_0^1\{1-4x+5x^2-x^4\}dx+c_1\int_0^1(-x+x^2)dx$$



Eg1. Extremize $F(y)=\int_0^1\left[\frac{1}{2}y'^2+\frac{y^2}{2}-y\right]dx$ subject to $y(0)=0=y(1)$

Note: E-L Eqns: $y''-y=-1$ (chk.)

Hom. sol^n: $y''-y=0$ $\leftarrow$ $y_h=Ae^x+Be^{-x}$

Part. ": $y_p=1$

$$y(x)=y_h+y_p=Ae^x+Be^{-x}+1$$

$$=\left(\frac{e^{-1}-1}{e-e^{-1}}\right)e^x+\left(\frac{1-e^{-1}}{e-e^{-1}}\right)e^{-x}+1$$

Using EFDM: ① Take $x_i=i/n$, $i=0,1,\ldots,n$

② Take $y_0=y_n=0$

③ Take $\Delta x=\frac{1-0}{n}=\frac{1}{n}$

Examples: find the extremal of F(y) $=\int_0^1\left[\frac{y'^2}{2}+\frac{y^2}{2}-y\right]dx$ with y(0) = y(1) = 0 . So, I know that my

Euler Lagrange method, my EL equation reduce to the following ODE for this problem and, well, let us go back to few slides to check what was the solution, we get the following ODE and now if we want so we are in a position that whatever solution that we get by the Ritz method, we can check it with the solution of the Euler Lagrange to see how close is it to the exact solution. So, if we were to use the Ritz method, so let us say this is the solution. Ritz method shows that this is $y_n(x) = \phi_o(x) + \sum_{i=1}^{n} C_i\phi_i(x)$.

We take $\phi_o(x) = 0$ and because by taking this value, it the y_n's immediately are going to satisfy the boundary condition and $\{\phi_i\}_{i=1}^{n} = x^i(1-x)^i$ So, this is what we choose in our basis function. It can be shown that these are linearly independent basis functions.

So, all we need are linearly independent basis functions. So, then let us look at a simple approximation. So, we are going to approximate y with $y_1 = C_1\phi_1 = C_1x(1-x)$

$$\Rightarrow F_1(C_1) = F(y_1) = \int_0^1 \left[\frac{C_1^2}{2}(1-2x)^2\frac{C_1^2}{2}x^2(1-x)^2 - C_1x(1-x)\right]dx$$

$$\frac{C_1^2}{2}\int_0^1 \left\{1 - 4x + 5x^2 - x^4\right\}dx + C_1\int_0^1 (-x+x^2)dx$$

then once we perform all the necessary integration, I am going to get the following result.

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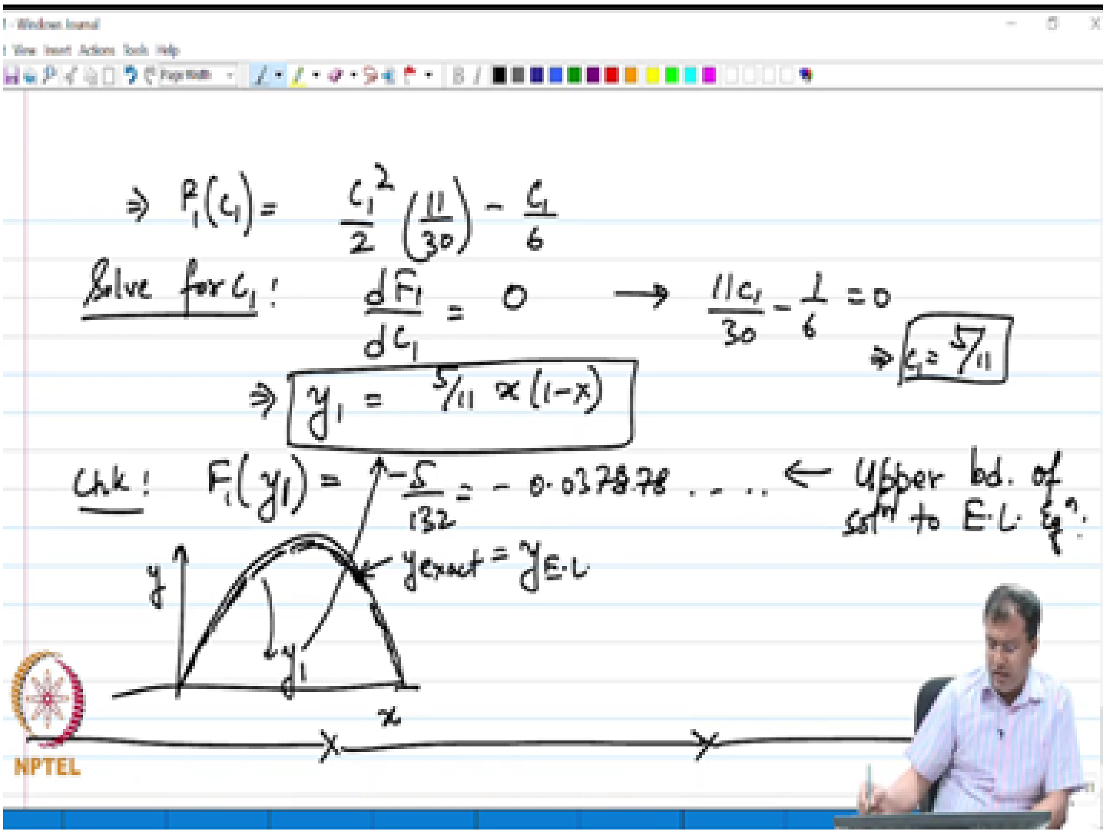

, $F_1(C_1) = \frac{C_1^2}{2}\left(\frac{11}{30}\right) - \frac{C_1}{6}$, we solve for C_1 we take $\frac{dF_1}{dC_1} = 0$, we used standard multivariate minimization argument or optimization argument.

$\Rightarrow \frac{11C_1}{30} - \frac{1}{6} = 0 \quad \Rightarrow \quad C_1 = \frac{5}{11}$

$\Rightarrow y_1 = \frac{5}{11}x(1-x)$

this is my approximate extremal that we have obtained using the Ritz method. We can check that the value of the functional evaluated at y_1 is coming out to be this following constant which is minus this quantity and by the analysis that I have just shown few slides back, this is the upper bound, this is the upper bound of the solution to the Euler Lagrange, the solution to the Euler Lagrange equation which was highlighted few slides back.

So, before we move on, I just want to highlight how does this approximate solution compares with the exact solution? So, if we were to plot y versus x, the exact solution is as follows. So, this is the y exact or this is also equal to the y obtained from the Euler Lagrange method and the solution that we obtained above which is my y_1 which is this particular solution is quite close to the exact solution.

So, even with this approximation with one basis function, it just close enough approximation to the exact solution. So, alternatively the choice of ϕ is ours. The choice of the basis function is ours and sometimes the choice is good while some other times the choice is poor. So, so let us look at another choice of the same function.

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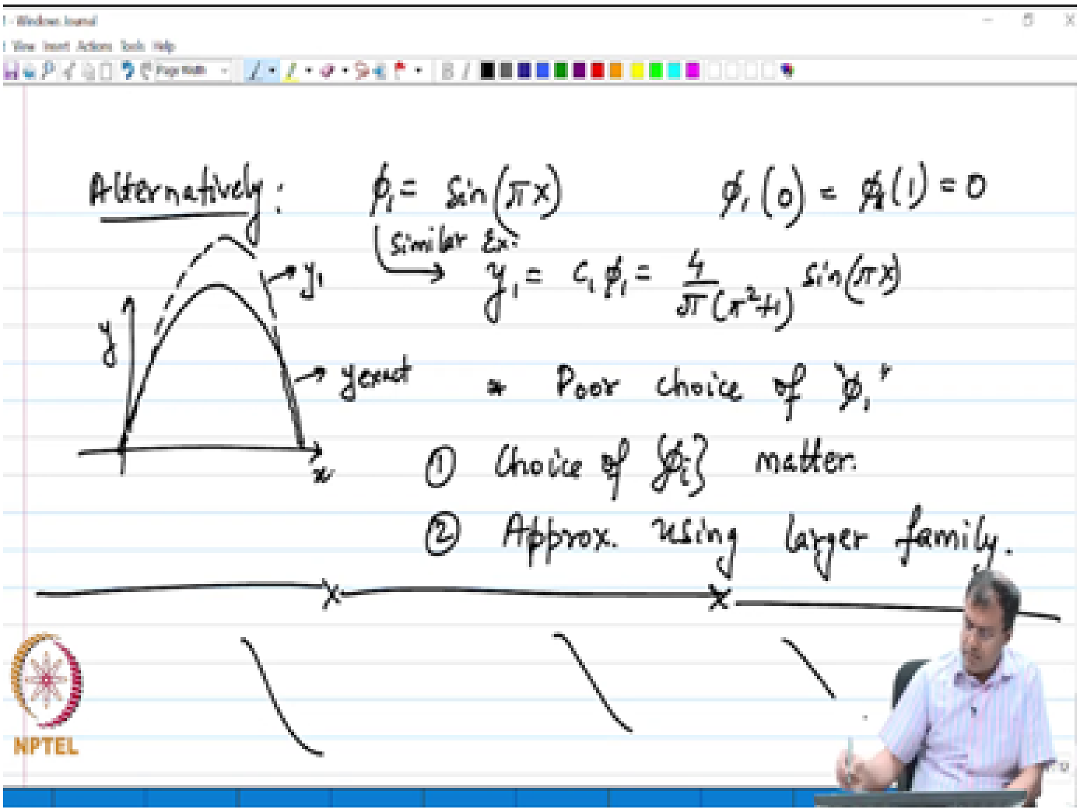

So, alternatively suppose we were to choice the following basis function. Suppose we were to choice $\phi_1 = \sin(\pi x)$. Notice that $\phi_1(0) = \phi_1(1) = 0$. So, a similar exercise of finding the approximate solution leads us to $y_1 = C_1\phi_1 = \frac{4}{\pi(\pi^2+1)} \sin \pi x$.

So, use the Ritz method and we will see that the solution comes out to be the following and if we were to now plot the exact solution with respect to let us say the approximate solutions. So, let us say this is my exact solution; the approximate solution in this case is poorer. So, which means in this case we have a poor choice of ϕ_1. So, this ϕ_1 is not a good choice.

So the moral of the exercise that we have done so far is that the choice of the basis function ϕ_i's matter. So, I am not going to go into more detail as to what sort of ϕ_i's will be suitable for what sort of functional but I am going to definitely give a reference which talks in depth about the choice of the Ritz basis function.

So, first of all the choice of ϕ_i's matter and also the approximation becomes better using larger families of ϕ_i's. That is this advice is irrespective of the method whether we use Euler, whether we use Ritz method or another method that we are going to describe shortly. So, use larger and larger family to get better approximation. So, let us look at an example.

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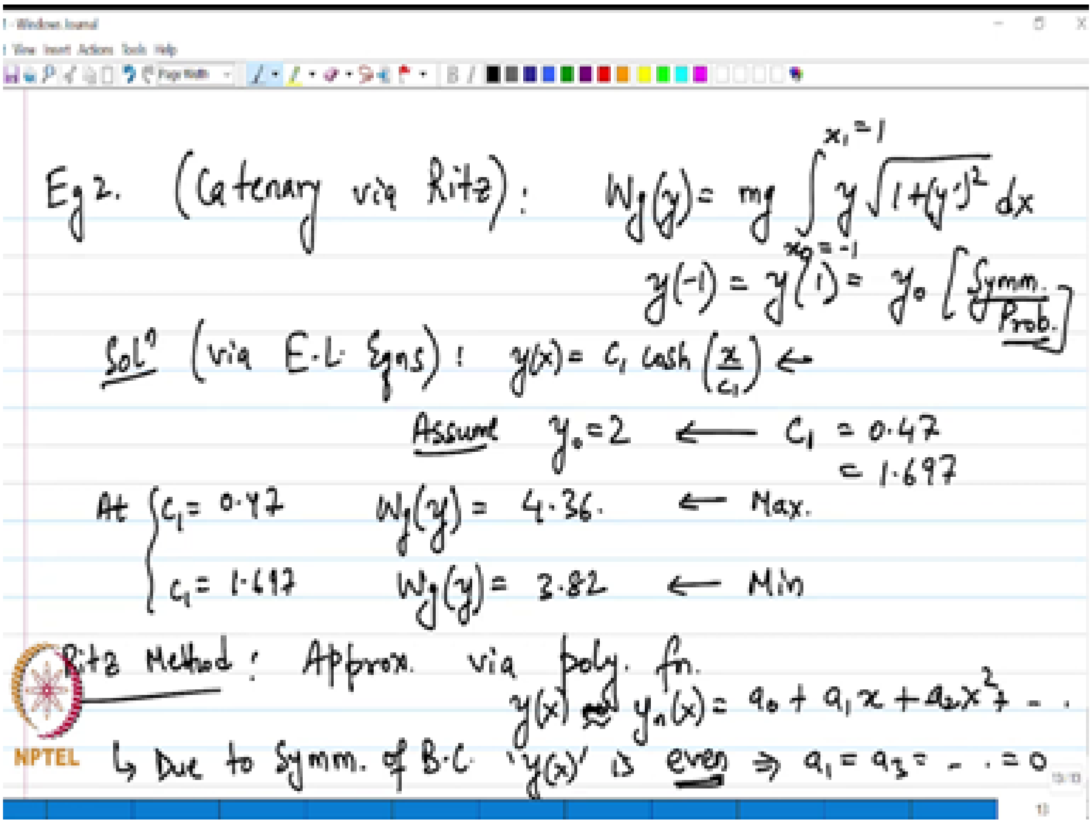

So, I am going to highlight an example. I am going to solve the same catenary problem via the Ritz method, my functional which is the potential energy functional in the catenary problem is as follows $W_g(y) = mg\int_{x_o=-1}^{x_1=1} y\sqrt{1+(y')^2}dx$ and with the boundary condition $y(-1) = y(1) = y_o$.

So what we have done, we are solving a symmetric problem, where the height of the extremal is the same at both the boundary points. So, we all know over the disclose of the last few lectures that the solution to the catenary problem via the Euler Lagrange equation is in the following form $y(x) = C_1 \cosh\left(\frac{x}{C_1}\right)$

let us assume further that our $y_o = 2$ So, $y(1)$ and y(-1) is equal to 2. We are assuming some value, some numerical value so that gives my 2 values of C_1. So, we already know the solution, when we impose the boundary condition, y(-1) = y(1) = 2 , we get two values of C_1, C_1 is either 0.47 or C_1 is also 1.697.

We have already discussed that if your y is above a certain critical value, I am going to get 2 solutions. Not necessarily both of them minima. So, in that case, we can immediately see which one is minima or which one is maxima. So, at the value c1 is equal to 0.47, I see that F_1 at C_1 the value of the functional F(y) comes out to be 4.36. So, we can just plugin C_1 and plug the extremal y here into the functional of our potential energy function.

and let me use the same rotation $W_g(y)$, it comes out to be this value and at C_1 equal to the other value, I see that my potential energy functional comes out to be the following quantity. So, this one is the minima and this one is the case of maxima. So, let us see what happens when we find the solution via the Ritz method. So, we are going to in our Ritz method, we are going to approximate via polynomial functions.

So, I am going to say that my y(x) $=\approx y_n(x) = a_o + a_1 x + a_2 x^2 + ...$ Now, before we move ahead, we also know that the problem is symmetric. We stated that to begin with which means that the solution that we are assuming must also be an even function to account for the symmetry.

which means due to symmetry of the boundary condition, y(x) is even, which means that my odd coefficients $a_1 = a_3 = ... = 0$

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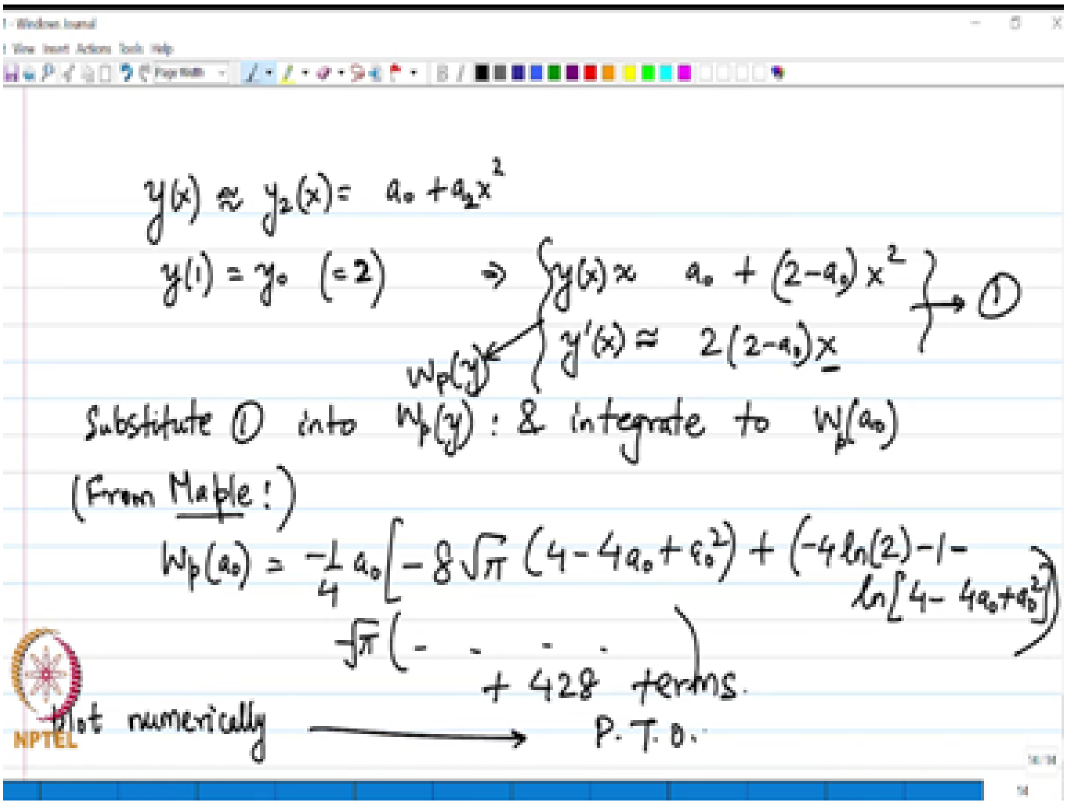

So, let us now approximate $y(x) \approx y_2(x) = a_o + a_2 x^2$ So, let us use the least order approximation and see how does the Ritz method compares? So, further we know that y(1) = $y_o = 2$ which we will use later on.

In fact, let me plug the value here. So, this means that y(x) now is, I can eliminate with this boundary condition, I can eliminate one of the constants a_2, so y(x) becomes $y(x) \approx a_o + (2 - a_o)x^2$, similarly I can find $y^{'}(x) \approx 2(2 - a_o)x$ **1**

So, then I substitute all these into my functional of $W_p(y)$, the functional that I had and find out the expression, So, we substitute **1** into our functional $W_p(y)$ and integrate. We can very clearly integrate with respect to the variable x.

The variable x can be integrated out and we are going to be left out with a function of 1 variable a_o. It is easier said than done because we will see that this function is very-very complicated. So, integrate to get $W_p(a_o)$. We see that, let me just show you that the function that is generated here by showing you some of the terms. So, these terms are generated from the software Maple.

So, Maple gives when we feed all these approximations into the functional in this software Maple, we see that my explanation $W_p(a_o)$ is the following. So, let me write few terms, in fact, I am never going to complete writing all the terms. So, let me just show what is the complexity that we have.

$$W_p(a_o) = -\frac{1}{4}a_o\left[-8\sqrt{\pi}(4-4a_o+a_o^2) + \left(-4\ln(2) - 1 - \ln\left(4-4a_o+a_o^2\right)\right) - \sqrt{\pi}(....) + 428 \text{ terms} \right]$$

So, notice that complexity that we have. So, what I am trying to show here is finding the optimum solution of W_p or finding the optimal value of a_o by hand is almost impossible right? We cannot just take that first derivative and second equal to zero and find the solution analytically.

However, if you feed this entire expression in the software itself and draw the figure $W_p(a_o)$of versus a_o, we can see where is the maximum or the minimum. So, when we do that, we plot numerically, let me show the plot in the next page.

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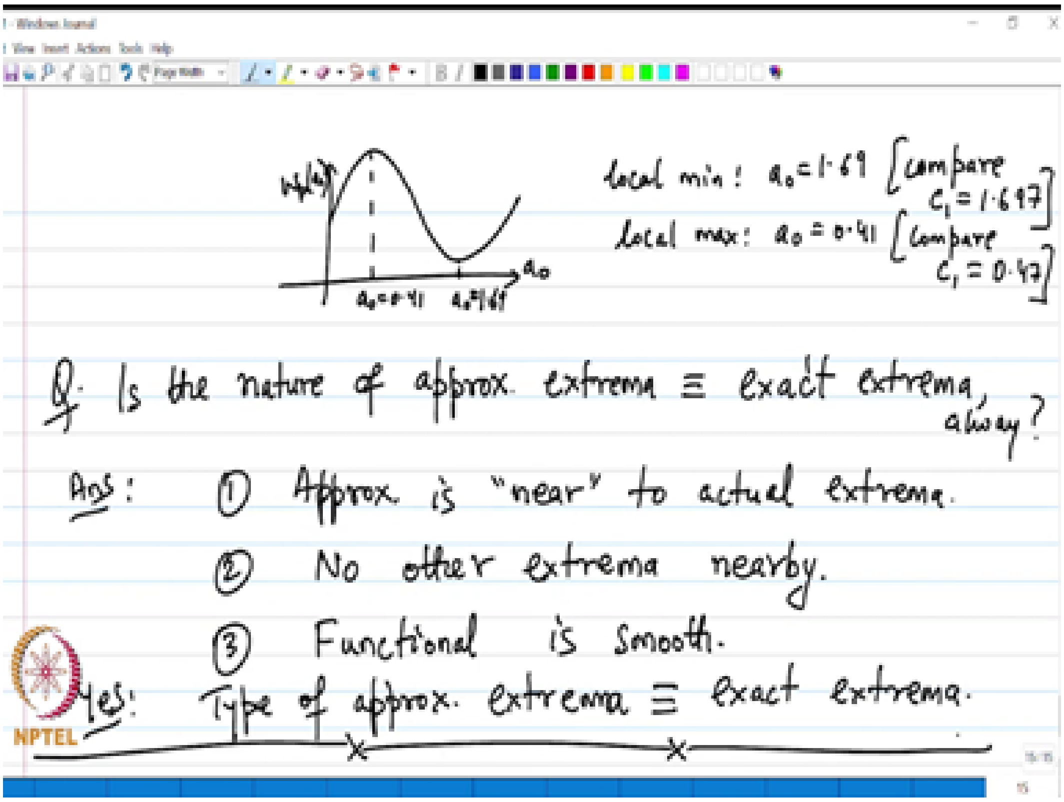

We plot numerically and highlight that $W_p(a_o)$of versus a_o. We see that we get the following function, you get the following curve where we obtain this maxima at $a_o = 0.41$ and the minima is obtained at $a_o = 1.69$, we can see that the local min is at a_o equal to point, 1.69 and compare the local min for C_1 which was at 1.697 for the exact function to the Euler Lagrange equation.

and the local max that we get here is $a_o = 0.41$ and again compare it with the local max at $C_1 = 0.47$. So, it seems that this case my minimum and the maximum are quite accurately captured by the Ritz method of the original extremal. So, this question is, so this question that we have to ask, when is this case always true or what I am asking is the following, what I am asking is the nature of the approximate extrema identically equal to the exact extrema.

So, we saw the exact solution, the solution to the Euler Lagrange and the solution obtained by the Ritz method and we saw that the maxima and the minima they are almost identically equal. So, this question is the nature of the approximate extrema equal to the exact extrema always, is this always true, the answer to this is depends on how good is the approximation.

So, the answer to this is if I have that approximation is near, when I say near the near is in terms of a certain norm. So, near to the actual extrema, if the approximate and further there is no other extrema, there is no other extrema nearby. So, we do not have multiple solutions to the Euler Lagrange equation. Otherwise, the approximation to the via the Ritz method might give erroneous answers and finally the functional that we are optimising is smooth so that the derivatives exist, then the answer is yes.

So, then yes that is the type of approximate extrema is identically equal to the exact extrema. So, that completes this particular example completes the discussion of the Ritz method. The students are asked, I am going to provide a reference towards the end and the students are asked to solve more examples via the reference as well as our homework modules and the tutorials.