Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 16
Generalization/Numerical Solution of Euler Lagrange Equations Part 4

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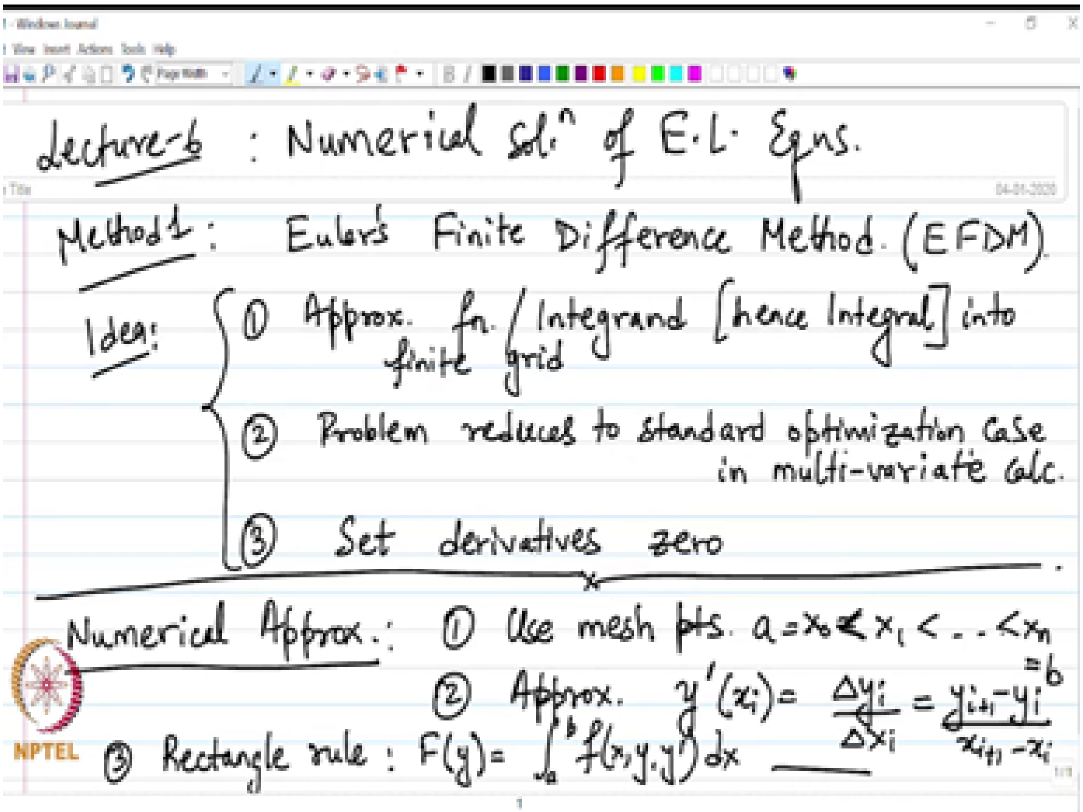

In today's lecture I am going to talk about the numerical solutions of Euler Lagrange equation. So, far we have discussed about the various forms of Euler Lagrange Equation and also we have found the solution to the Euler Lagrange Equation purely using analytical techniques. So, in today's lecture, I am going to talk about what happens when we are stuck in a situation when the Euler Lagrange Equation are no longer solvable analytically.

So, in that case we take the help of numerics and I am going to discuss 3 different methods today in this lecture on how to solve and later on in this, the course of this lecture, I am also going to provide some other references for people interested in developing higher order numerics. So, the first method that we will talk about in this class today is regarding the Euler's Finite Difference method.

So, this seems to be the simplest method that we can adopt in finding the solutions of the Euler Lagrange Equations numerically. Let me abbreviate this method by EFDM. So, the basic idea of this method is as follows: the idea is, we are going to approximate our function or the function which involved in this functional optimization is the integrant.

so we approximate the integrant and hence the integral into finite grids points. So, instead of now solving the, finding the optimum value, the optimum function of an integral, we are now going to find the optimum value of a summation. So, the integral in the functional changes to a summation over the particular set of grid points.

The moment we discretise our functional the integral into finite set of points, our integral becomes a summation and then the problem reduces to the standard optimization problem. The problem reduces to a standard optimization case in multivariate calculus. So, the way we approach maximizing or minimizing a function in regular vector space that is the same way that we adopt in this strategy.

and finally the way to do it is, after we have discretised our functional, all we have to do is to find the optimal solution by setting the derivatives equal to 0, the derivatives with respect to the unknowns equal to 0. So, these 3 points reflect the basic idea of the numerical solution. Now, let me just build a little bit more background before I am going to highlight this methodology using some examples.

let me just say that, we have a finite set of points, so the way we numerically approximate our problem is that we use finite set of mesh points $a = x_o < x_1 <, ... < x_n = b$

So, our integral is from a to b. And we approximate our derivative $y^{'}(x_i) = \frac{\Delta y_i}{\Delta x_i} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$, that is the Euler's Forward Difference Approximation that we are using to approximate the derivatives at the points of our points.

Finally the third rule involves, we use the standard rectangle rule to change our integral into summation $F(y) = \int_a^b f(x, y, y^{'})dx$ and we change it using into a summation which is as follows

$$= \sum_{i=0}^{n-1} f(x_i, y_i, \frac{\Delta y_i}{\Delta x_i})\Delta x_i = \bar{F}(\bar{y}) = \bar{F}(y_o, y_1, ..., y_n)$$

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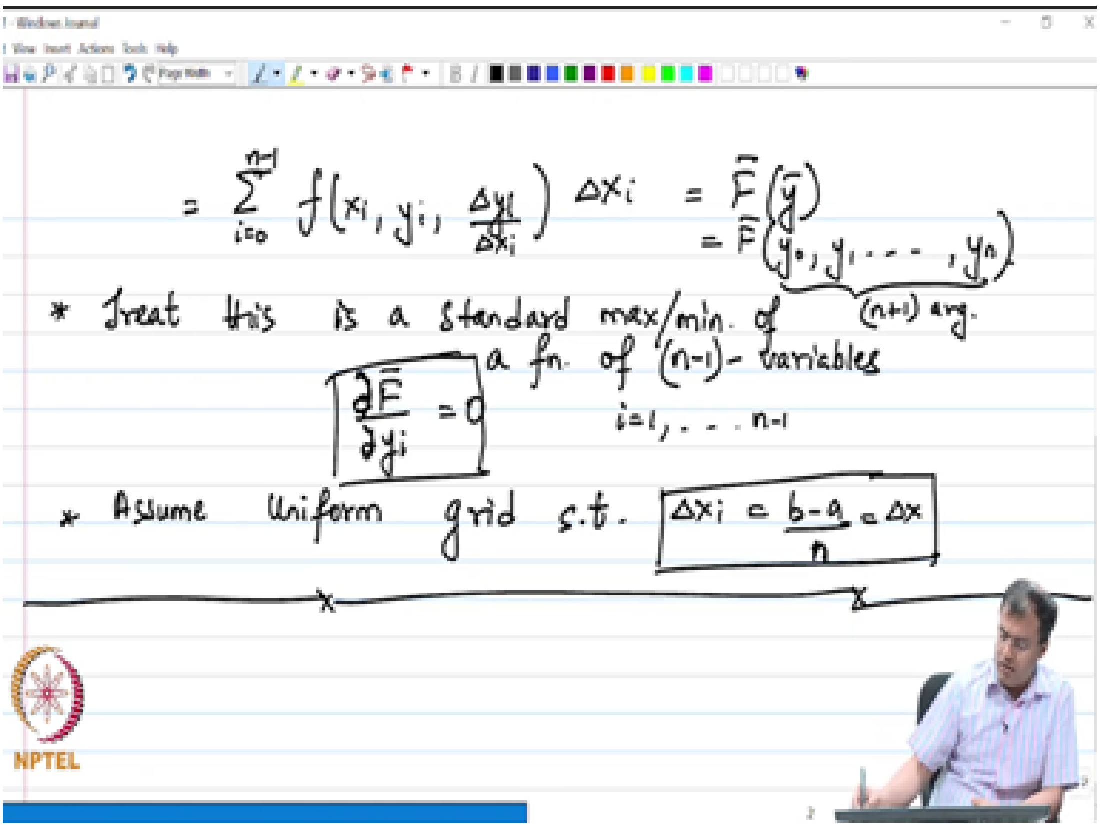

The idea is to treat the maximization of this function using the standard multivariate calculus. So, before we do that, let us notice that this particular function has (n + 1) arguments.

However, we have 2 arguments corresponding to the boundary condition, so y_o and y_n are the values of y at x_o and x_n given by the corresponding boundary condition. So, there are (n - 1) unknown, this as a standard maximization problem. Minimization of a function of (n - 1) variables. The two of the variables are just the boundary conditions.

We find the solution to this problem by taking the derivative of f with respect to the unknowns $\frac{\partial \bar{F}}{\partial y_i} = 0, i = 1, 2, ...(n-1)$, this is the set of equations that we solve to find the extremum of this function.

Further let us assume that the grids are uniform, right now we are going to make this assumption so as to simplify our calculations, assume uniform grid such that $\Delta x_i = \frac{b-a}{n} = \Delta x$, that is required in this method is to just find the maximum of this function using the standard first derivative test.

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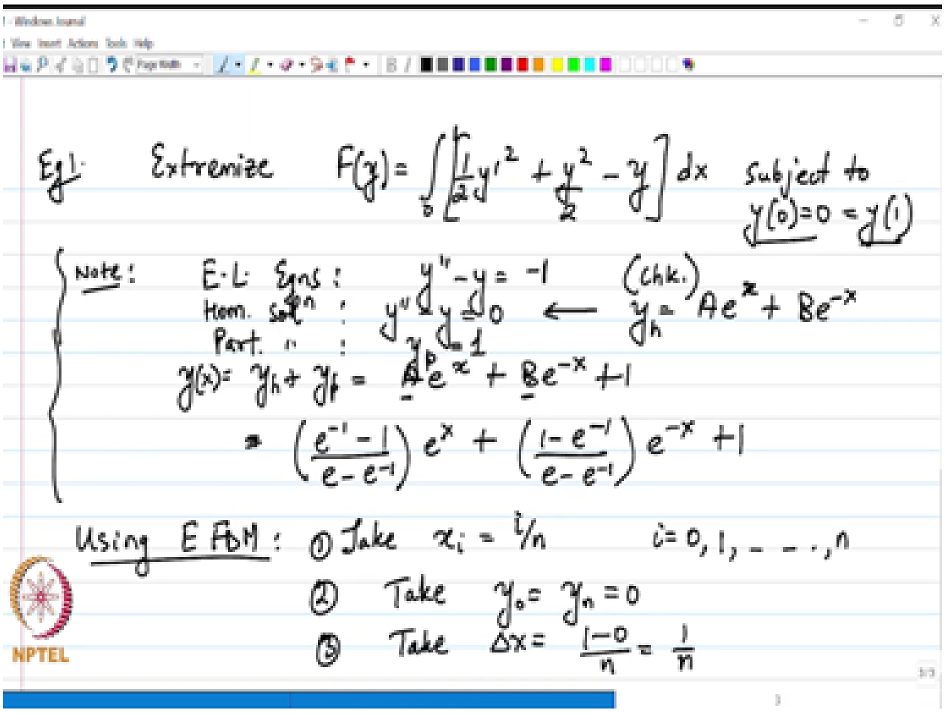

Example 1: To find the extremal $F(y) = \int_0^1 \left[\frac{y'^2}{2} + \frac{y^2}{2} - y\right] dx$ subject to the boundary condition

y(0) = 0 = y(1), we find the extremum of this function using the Euler's Finite Difference method let us see what is the exact solution given by the Euler Lagrange method, the extremum y is given by the Euler Lagrange equation which reduces to $y'' - y = -1$

Homogenous solution of $y'' - y$ is y = $Ae^x + Be^{-x}$ and the particular solution is given by $y_p = 1$ So, y(x) $= y_c + y_p = Ae^x + Be^{-x} + 1$, we have 2 unknowns A and B and they can be found using these 2 boundary conditions and let me just write down the final answer after finding the constants

$$y(x) = \left(\frac{e^{-1} - 1}{e - e^{-1}}\right)e^x + \left(\frac{1 - e^{-1}}{e - e^{-1}}\right)e^{-x} + 1$$

now we are going to discretise our problem, our functional and see that Euler Langrage method all is written, is well approximated by the Euler Finite Difference method or not. So, let us look what happens when we use the finite difference method. let us as shown in the previous slide let us take a uniform grade.

We take **1** $x_i = \frac{1}{n}$, i = 0,1,.....n, Take $y_o = y_n = 0$ and Take $\Delta x = \frac{1}{n}$

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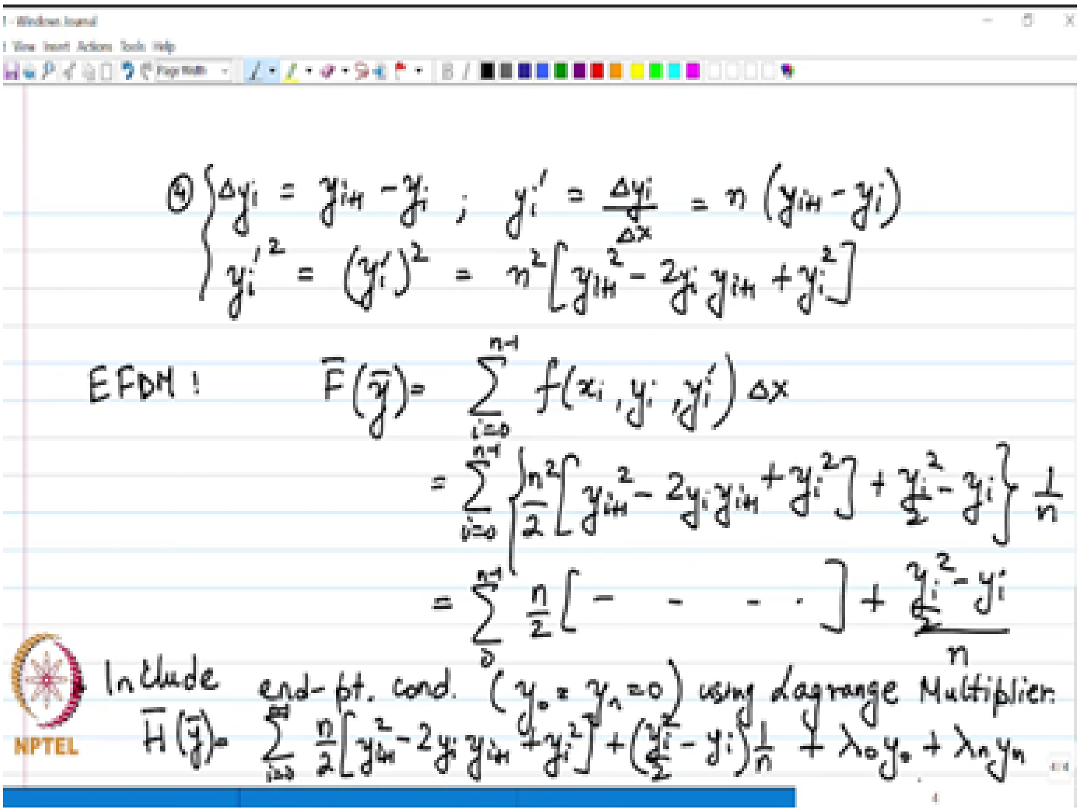

$$\Delta y_i = y_{i+1} - y_i; \quad y_i^{'} = \frac{\Delta y_i}{\Delta x} = n(y_{i+1} - y_i)$$

$$y_i^{'2} = (y_i^{'})^2 = n^2 \left[y_{i+1}^2 - 2y_i y_{i+1} + y_i^2 \right]$$

So, we are ready to substuite all this values in our original functional. So, the Euler's Finite Difference method reduces to the following function. We see that my vector function is as follows

$$\bar{F}(\bar{y}) = \sum_{i=0}^{n-1} f(x_i, y_i, y_i^{'})\Delta x = \sum_{i=0}^{n-1} \left\{ \frac{n^2}{2} \left[y_{i+1}^2 - 2y_i y_{i+1} + y_i^2 \right] + \frac{y_i^2}{2} - y_i \right\} \frac{1}{n}$$

So, what we can do is we can further include the two unknowns coming from the boundary that is y_o and y_n we are going to include the end points condition, they are already 0, we include them as a Lagrange condition.

so students who have done standard multivariate calculus should be aware of the Lagrange Multiplier method, if they are not we are going to a very brief revision very soon. So, using the Lagrange Multiplier method we write down the new vector function.

$$\bar{H}(\bar{y}) = \sum_{i=o}^{n-1} \frac{n}{2}\left[y_{i+1}^2 - 2y_i y_{i+1} + y_i^2\right] + \left(\frac{y_i^2}{2} - y_i\right)\frac{1}{n} + \lambda_o y_o + \lambda_n y_n$$

we have included those using these additional constants known as the Lagrange Multiplier, the next step to the solution involves, we take the derivative of this vector function with respect to unknowns and set them equal to 0, since we have included even y_o and y_n we assume that they are unknown, but later own we are going to use a boundary condition to eliminate them. So, if we were to assume that y_o and y_n are unknown we now have n + 1 unknowns, in fact n + 3 unknowns including λ_o and λ_n.

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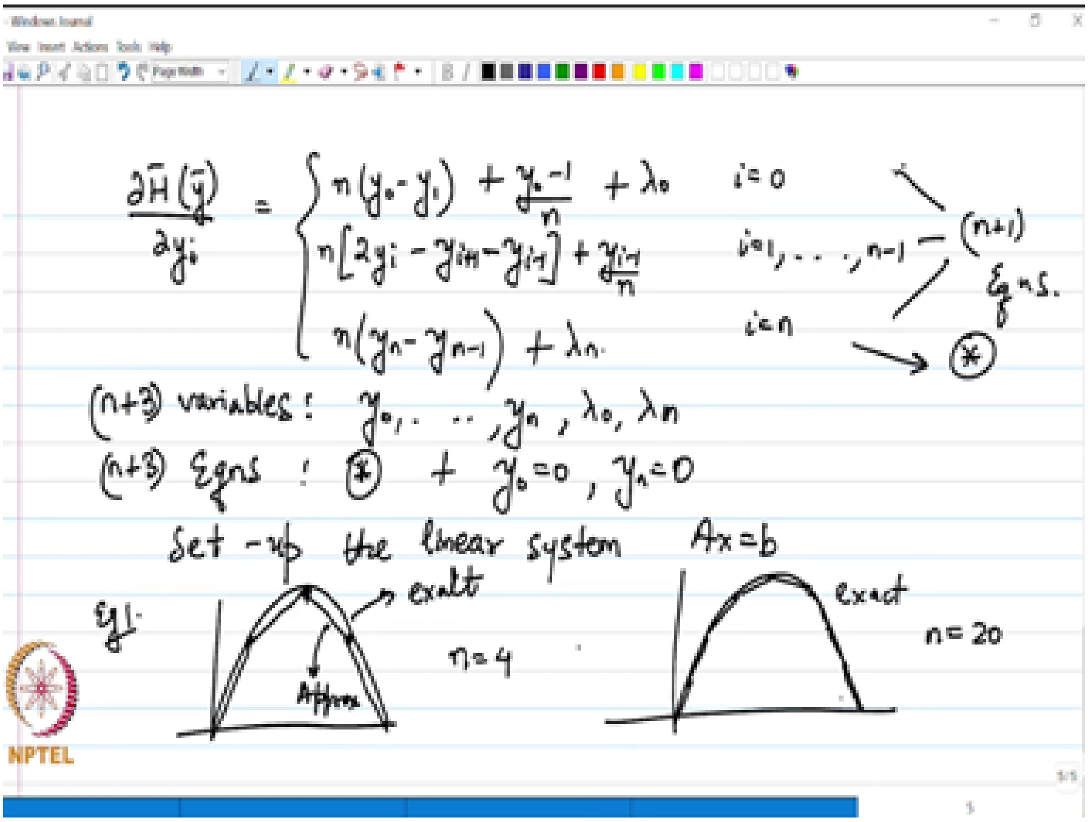

$$\frac{\partial \bar{H}(\bar{y})}{\partial y_i} = \begin{cases} n(y_o - y_1) + \frac{y_o^{-1}}{n} + \lambda_o & i = 0 \\ n[2y_i - y_{i+1} - y_{i-1}] + \frac{y_{i-1}}{n} & i = 1, ...n-1 \\ n(y_n - y_{n-1}) + \lambda_n & i = n \end{cases} \qquad *$$

we have n + 3 variables and those variables are from y_o to y_n and the constants $\lambda_o, \lambda_1, \lambda_n$ and n + 3 equations

* is giving us n + 1 equations plus the fact that $y_o = y_n = 0$ coming from the boundary gives us the additional 2 equations to solve. So, the system is fully solvable. The number of unknowns are equal to the number of equations and we should get a unique solution.

So, then all I have to do is to set up the linear system Ax = b, we solve for the system that gives us the unknowns $y_i's$ and those $y_i's$ are the appropriate solution to the extremal and that is the end of the solution methodology, but just to highlight how this solution compares with our exact solution. We have done, I have done some numerical simulations and I would like to compare how does it matches with the exact solution.

In this case, this is our example 1. So, if we were to plot the exact solution, exact solution was shown in our previous slide. The exact solution looks like this and if we take only 4 grid points, we see that 3 of them will be interior, there will be 2 which are on the boundary and we get a very gagged type of an appropriate solution. So, the approximate solution satisfies the exact value of the functional, only at 5 different points including 2 at the boundary.

However, if we do the same exercise comparing with the exact solution with let us say higher number of grid points. Let us say n equal to 20, we see that the solution is much more smooth and closer to the exact solution. So, that has all being, that is just what I am showing what we have got through the numerical output. So, the moral of the story here is the finer the grid point, the closer is the Euler's Finite Different solution to the exact solution.

In fact, we will show now next that the Euler solution converges to the solution given by the Euler Lagrange Equation in the $\lim_{\Delta x \to 0}$ or in the limit that the number of grid points approaches infinity.

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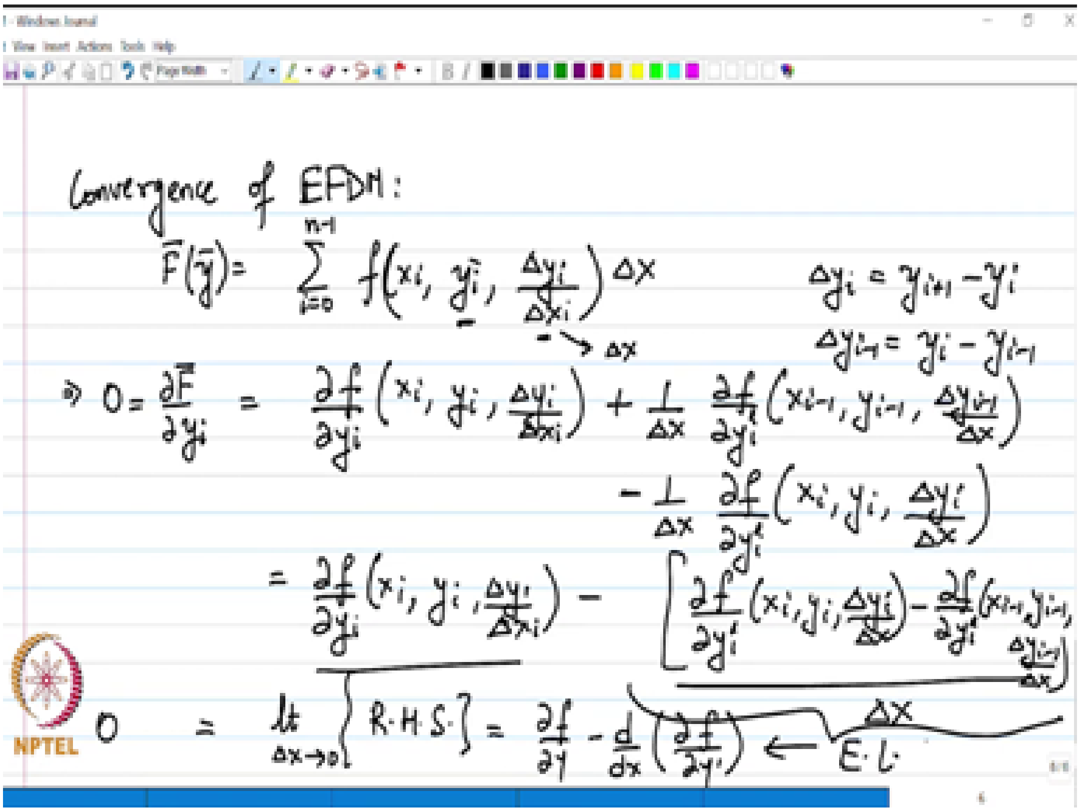

So what we are showing is the convergence of the Euler's Finite Difference method. Our vector function is as follows $\bar{F}(\bar{y}) = \sum_{i=0}^{n-1} f(x_i, y_i, \frac{\Delta y_i}{\Delta x_i})\Delta x_i$ Now we want to differentiate, the Euler's method differentiates the vector function with respect to y.

$$\Rightarrow 0 = \frac{\partial \bar{F}}{\partial y_i} = \frac{\partial f}{\partial y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) + \frac{1}{\Delta x}\frac{\partial f}{\partial y_i}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right) - \frac{1}{\Delta x}\frac{\partial f}{\partial y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right)$$

$$= \frac{\partial f}{\partial y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) - \frac{\left[\frac{\partial f}{\partial y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\partial f}{\partial y_i}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)\right]}{\Delta x}$$

Now take the $\lim_{\Delta x \to 0}$ on this right hand side involves this term as well as this whole term and we see that this quantity is nothing but

$$\Rightarrow \qquad 0 = \lim_{\Delta x \to 0} \{\text{R.H.S}\} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

we see that this is nothing but the Euler Lagrange equation. So, what I have shown here is in the limit the Euler's Finite Difference method gives a solution which approaches the solution of the Euler Lagrange method.