Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 15

Generalization/Numerical Solution of Euler Lagrange Equations Part 3

let us now look at another case of generalization of the Euler Lagrange, namely the case containing more than one independent variable and let us restrict our attention to this case having just one dependent variable.

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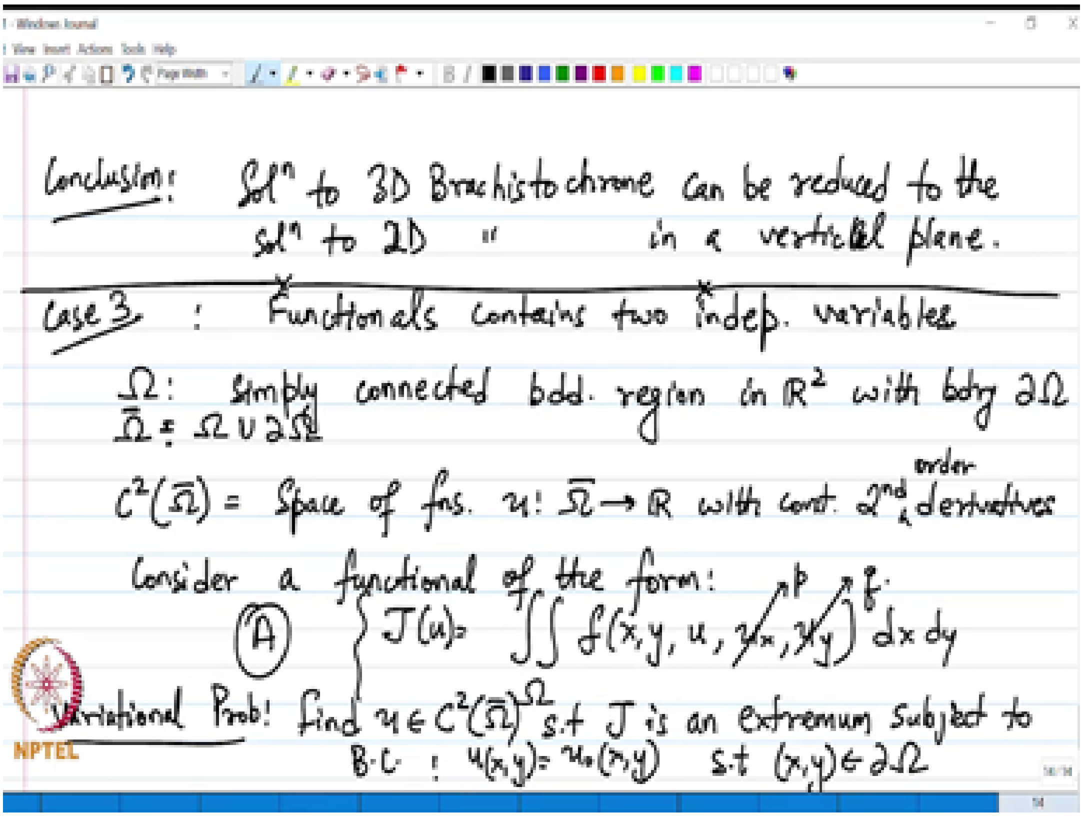

this is a case where the functional contains two independent variables, we will see that this case is significantly harder to solve. So, let us develop the theory in this case using an arbitrary domain formulation and then we are going to restrict that domain into R^2. So, let us now say that we have a domain Ω where Ω is a simply connected domain, students can look up in Google as to what I mean by simply connected, should be familiar with this terminology.

Essentially I am saying that this is a domain which does not have any pathological problems. So, Ω is a simply connected bounded region in R^2 with boundary $\delta\Omega$ and we have that $\bar{\Omega} = \Omega \cup \delta\Omega$, this is the closure of the set omega. Then we further describe $C^2(\bar{\Omega})$ is the set of all continuously second order differentiable functions in $\bar{\Omega}$

Now we are ready to describe the functional in this case, we consider a functional of the form
J(u) = $\iint f(x, y, u, u_x, u_y)dxdy$ **A**

I call this variable u_x as p and u_y variable as q. The variational problem says we need to find u $\in C^2(\bar{\Omega})$ such that J is an extremum subject to the boundary condition given by u(x, y) = $u_o(x, y)$ s.t $(x, y) \in \partial\Omega$ then I also have to describe the perturbation to such class of functions. So, let us describe the perturbation.

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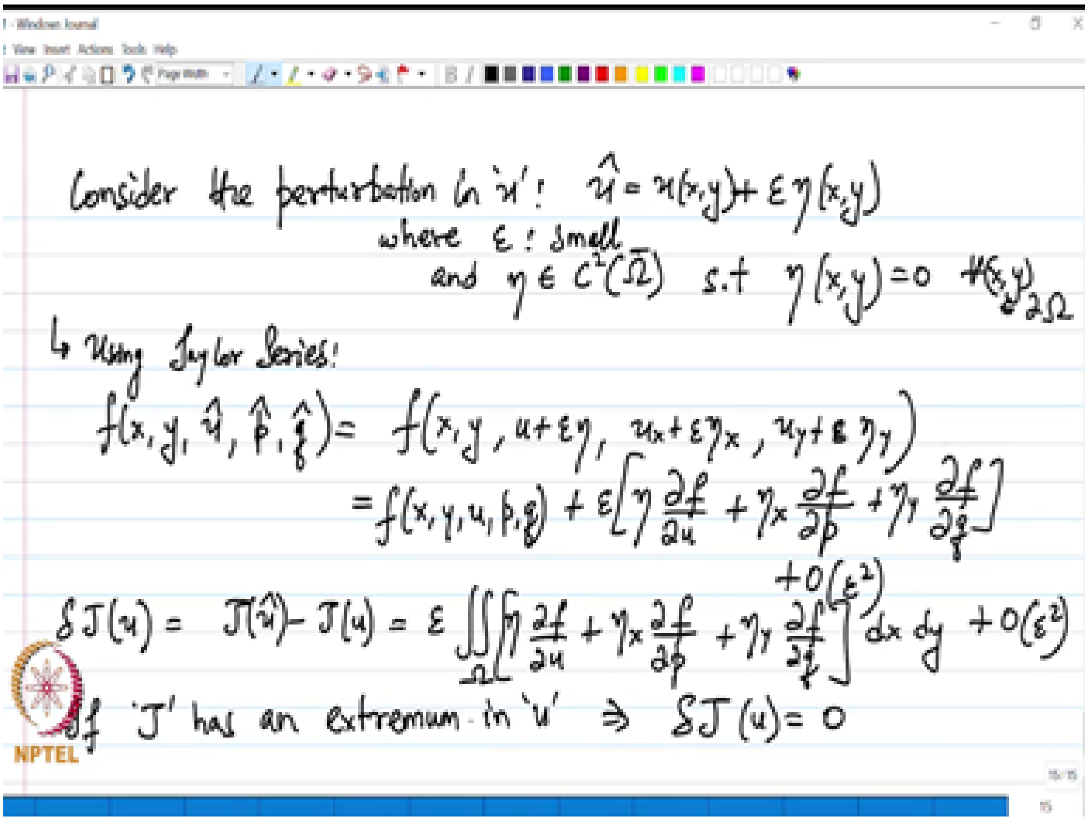

Consider the perturbation in u, which is going to be $\hat{u} = u(x, y) + \epsilon\eta(x, y)$ where ϵ is small and $\eta \in C^2(\bar{\Omega})$ s.t $\eta(x, y) = 0 \; \forall \; (x, y) \in \partial\Omega$

Using Taylor Series, we are ready to expand our integrant for the functional in terms of the extremal u

$$f(x, y, \hat{u}, \hat{p}, \hat{q}) = f(x, y, u + \epsilon\eta, u_x + \epsilon\eta_x, u_y + \epsilon\eta_y)$$

$$= f(x, y, u, p, q) + \epsilon \left[\eta \frac{\partial f}{\partial u} + \eta_x \frac{\partial f}{\partial p} + \eta_y \frac{\partial f}{\partial q} \right]$$

So, this has been expanded now up to the first order, we are trying to figure out the Euler Lagrange equation for this two independent variable case, so we have right now written the integrant in terms of using the Taylor Series, we have expanded the integrant, so then finally my variation in the functional J

$$\delta J(u) = J(\hat{u}) - J(u) = \epsilon \iint_{\Omega} \left[\eta \frac{\partial f}{\partial u} + \eta_x \frac{\partial f}{\partial p} + \eta_y \frac{\partial f}{\partial q} \right] dx dy + O(\epsilon^2)$$

Now, we have written the variation in terms of this double integral and we know that if J has an extremum in u $\Rightarrow \delta J(u) = 0$ So, then again the goal is to change this integral constraint into a differential constraint and to do that we have to utilize the so called Green's Theorem, because we are now working on an arbitrary domain Ω.

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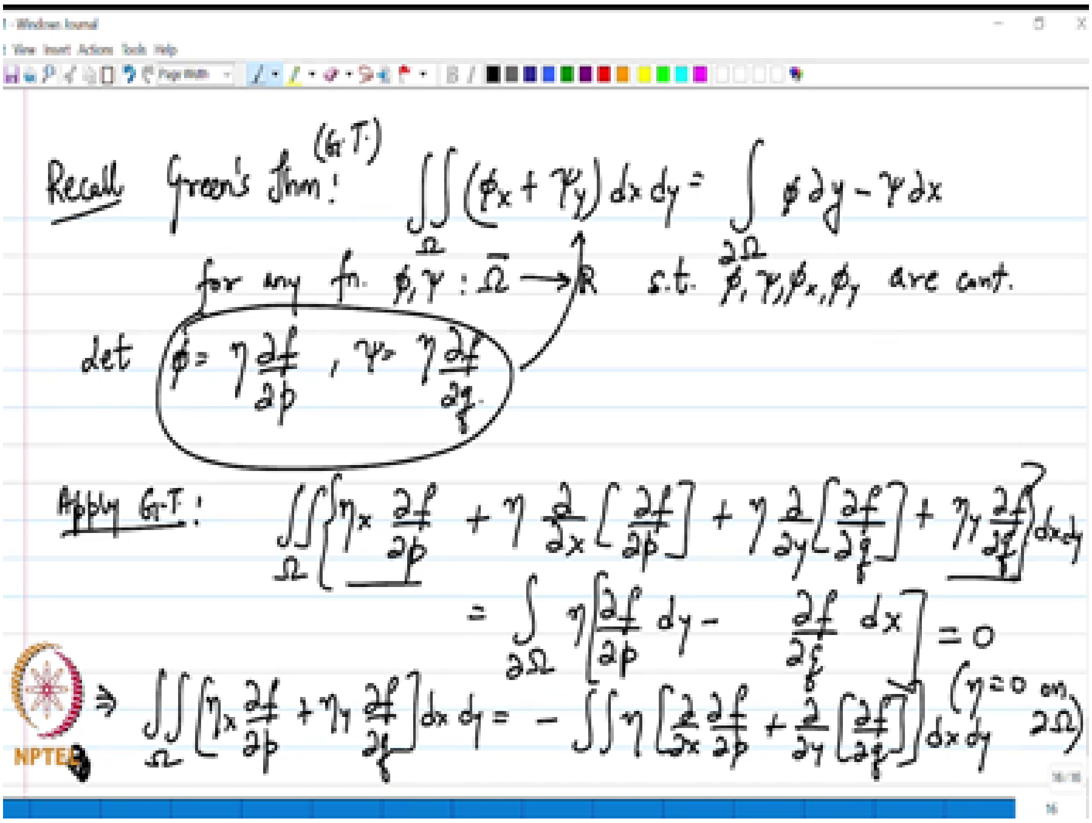

Green's Theorem in 2D says that

$$\iint_{\Omega} (\phi_x + \psi_y) dx dy = \int_{\partial\Omega} \phi \partial y - \psi \partial x$$

For any function $\phi, \psi : \Omega \to R$ s.t $\phi, \psi, \phi_x, \psi_y$ are all continuous.

$$\text{Let} \qquad \phi = \eta \frac{\partial f}{\partial p}, \psi = \eta \frac{\partial f}{\partial q}$$

Applying Green Theorem

$$\iint_{\Omega} \left\{ \eta_x \frac{\partial f}{\partial p} + \eta \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial p} \right] + \eta \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial q} \right] + \eta_y \frac{\partial f}{\partial q} \right\} = \int_{\partial\Omega} \eta \left[\frac{\partial f}{\partial p} dy - \frac{\partial f}{\partial q} dx \right] = 0$$

$$\Rightarrow \iint_{\Omega} \left\{ \eta_x \frac{\partial f}{\partial p} + \eta_y \frac{\partial f}{\partial q} \right\} dx dy = - \iint \eta \left[\frac{\partial}{\partial x} \frac{\partial f}{\partial p} + \frac{\partial}{\partial y} \frac{\partial f}{\partial q} \right] dx dy$$

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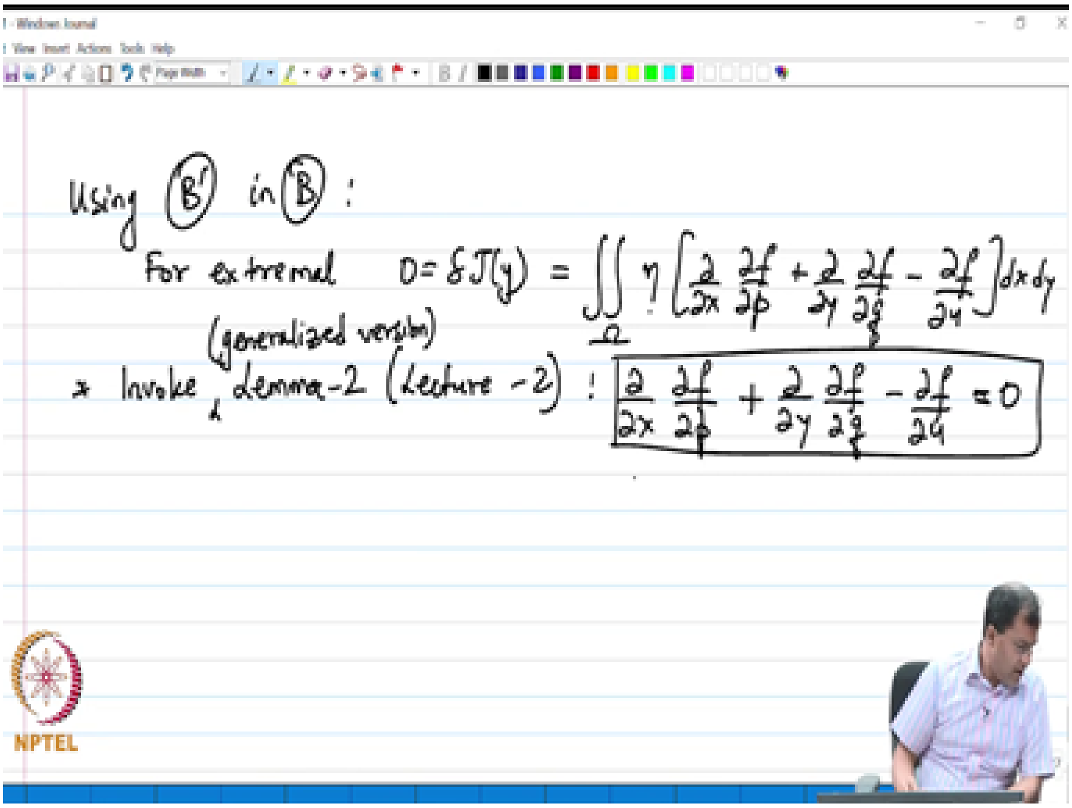

Using $B^{'}$ in B

$$\text{For extremal } \; 0 = \delta J(y) = \iint_{\Omega} \eta \left[\frac{\partial}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial}{\partial y}\frac{\partial f}{\partial q} - \frac{\partial f}{\partial u} \right] dxdy$$

Now we have been able to successfully separate out the perturbation function η and the next step involves invoking a generalized version of lemma 2 discussed in our lecture 2 to change this differential, this integral constraint into a differential constraint.

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial}{\partial y}\frac{\partial f}{\partial q} - \frac{\partial f}{\partial u} = 0$$

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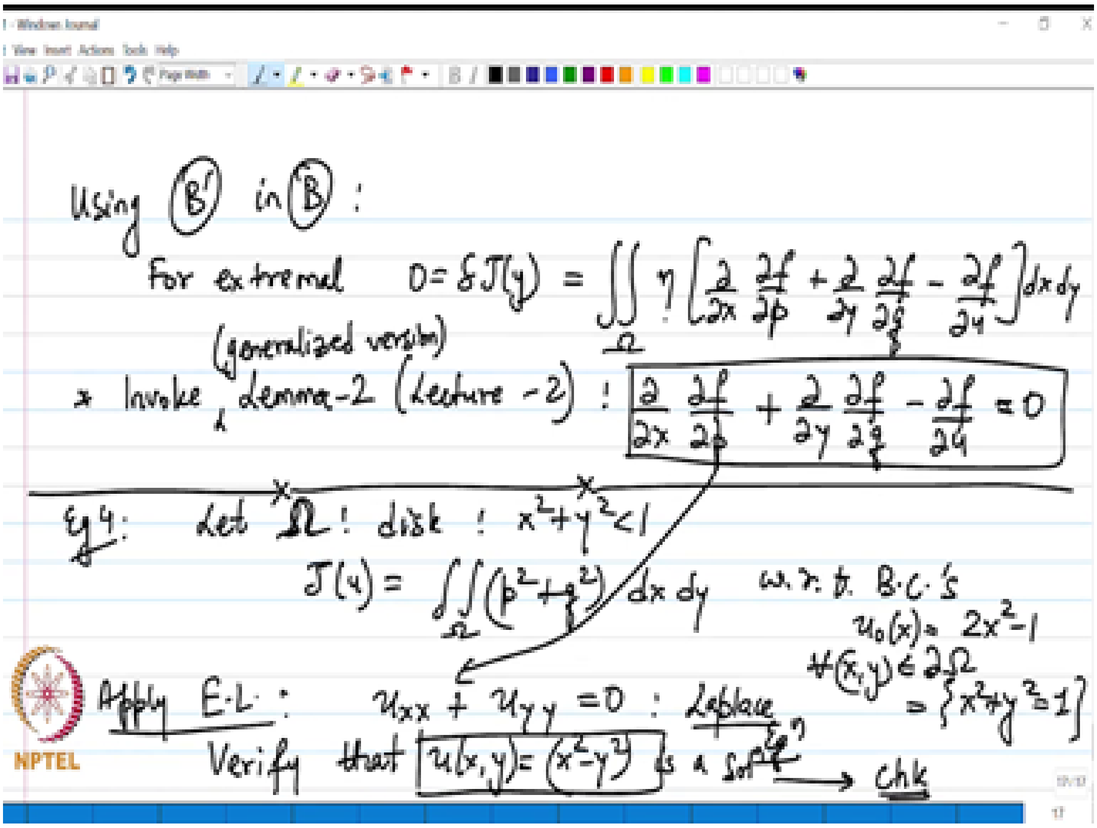

let us now look at an example in this case. let Ω domain be the disk which is given by $x^2 + y^2 < 1$, it is a unit disk and let us say functional $J(u) = \iint_\Omega (p^2 + q^2)dxdy$ where p and q are the derivatives of u with respect to x and y, with respect to the boundary condition $u_o(x) = 2x^2 - 1 \ \ \forall\ (x, y) \in \partial\Omega = \{x^2 + y^2 = 1\}$

When we apply the Euler Lagrange equation, we see that we are going to get that the extremal satisfies the following partial differential equation $u_{xx} + u_{yy} = 0$which is nothing but the Laplace equation.

The solution is not very short, it is a very lengthy exercise to solve this Laplace equation with a given boundary condition but students are asked to verify that the following satisfied the Laplace equation with the boundary condition, verify that $u(x, y) = x^2 - y^2$ is a solution to this problem.

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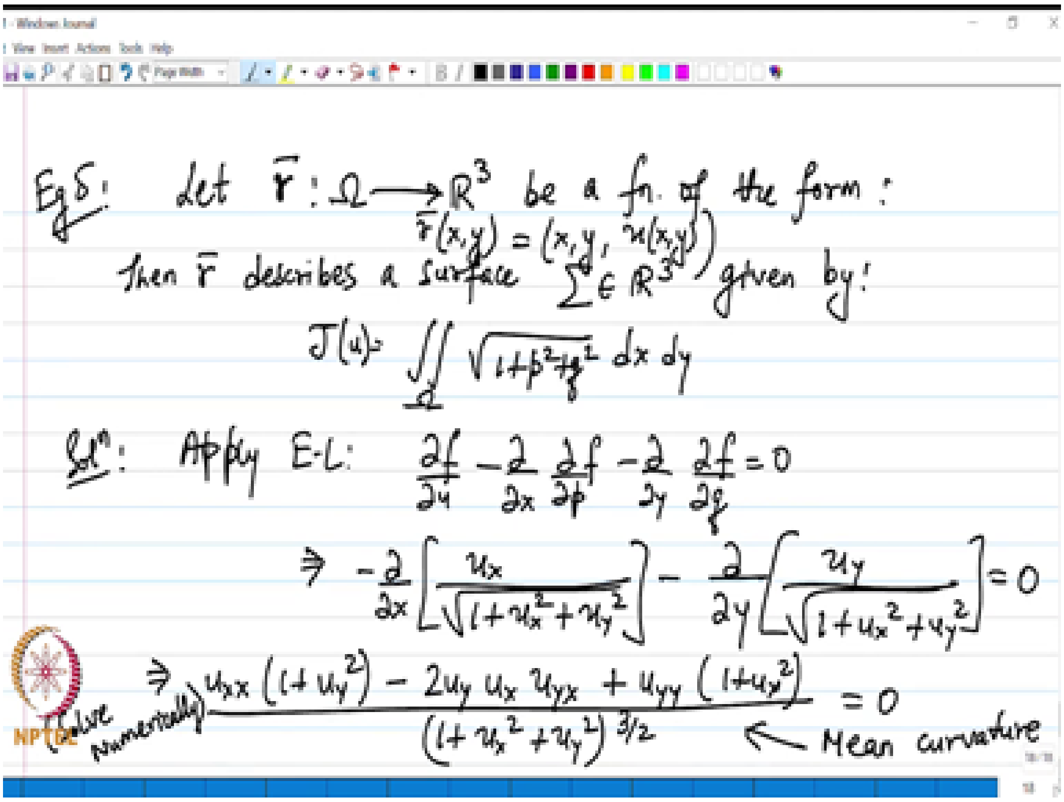

Then let us look at another scenario. We have another example here, let us describe a curve $\bar{\gamma} : \Omega \to \mathcal{R}^3$ be a function of the form $\bar{\gamma}(x, y) = (x, y.U(x, y))$. Then $\bar{\gamma}$ describes a surface $\sum \in \mathcal{R}^3$ given by

$$J(u) = \iint_\Omega \sqrt{1 + p^2 + q^2} dxdy$$

We apply the Euler Lagrange equation

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial}{\partial y}\frac{\partial f}{\partial q} - \frac{\partial f}{\partial u} = 0$$

$$\Rightarrow -\frac{\partial}{\partial x}\left[\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right] - \frac{\partial}{\partial y}\left[\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right] = 0$$

$$\Rightarrow \frac{u_{xx}(1 + u_y^2) - 2u_y u_x u_{yx} + u_{yy}(1 + u_x^2)}{(1 + u_x^2 + u_y^2)^{\frac{3}{2}}} = 0$$

To find the extremal to this problem we have to solve this monsters' PDE but, however, notice that people who are familiar with surface kinetics notice that this particular equation is nothing but the mean curvature of the surface in 3D.

So, which means so the expression on the left hand side is nothing but mean curvature, so the solution is such that the mean curvature has to be 0 and we cannot solve analytically at this stage, this equation has to be solved numerically. So, I am going to end my discussion in this lecture by mentioning the following issues.

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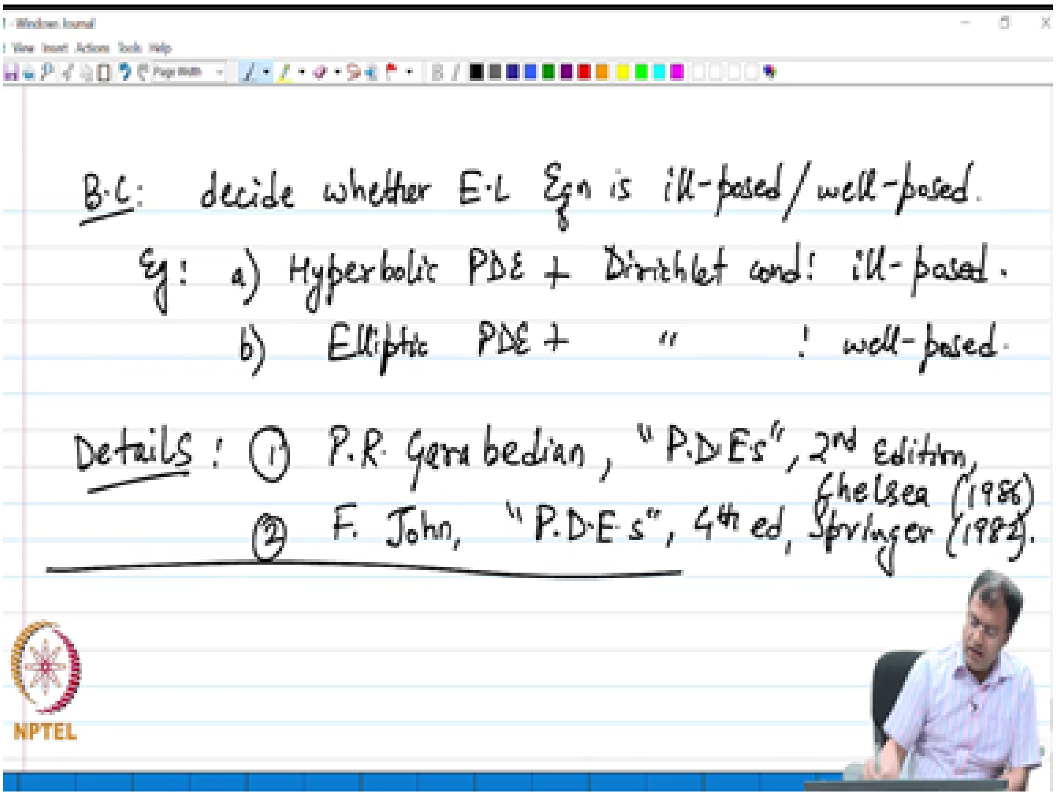

So far we have looked at two different cases, the extensions of Euler Lagrange, namely the extensions containing functions of several dependent variables and extensions containing function of several independent variables. It is also that the boundary condition also decide the fate of the solution of Euler Lagrange, in particular the boundary conditions are going to decide whether, they are going to decide whether the Euler Lagrange equation is ill-posed or well-posed.

For example, if we have a hyperbolic Partial Differential Equation along with the Dirichlet condition more often than not this is an ill-posed problem and on the other hand if we an Elliptic Partial Differential Equation or an Elliptic PDE along with the same Dirichlet boundary condition more often than not, this is a well-posed problem.

Students are asked to look at more details related to the boundary condition in this following to books

1 P. R. Garabeian, a Spanish author, which is on PDEs, second Edition, Chelsea, published in 1986
2 F. John on PDEs, fourth Edition Springer, published in 1982