Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 13
Generalization/Numerical Solution of Euler Lagrange Equations Part 1

Today in this lecture I am going to continue our discussion on the Generalization of Euler Lagrange Equations.

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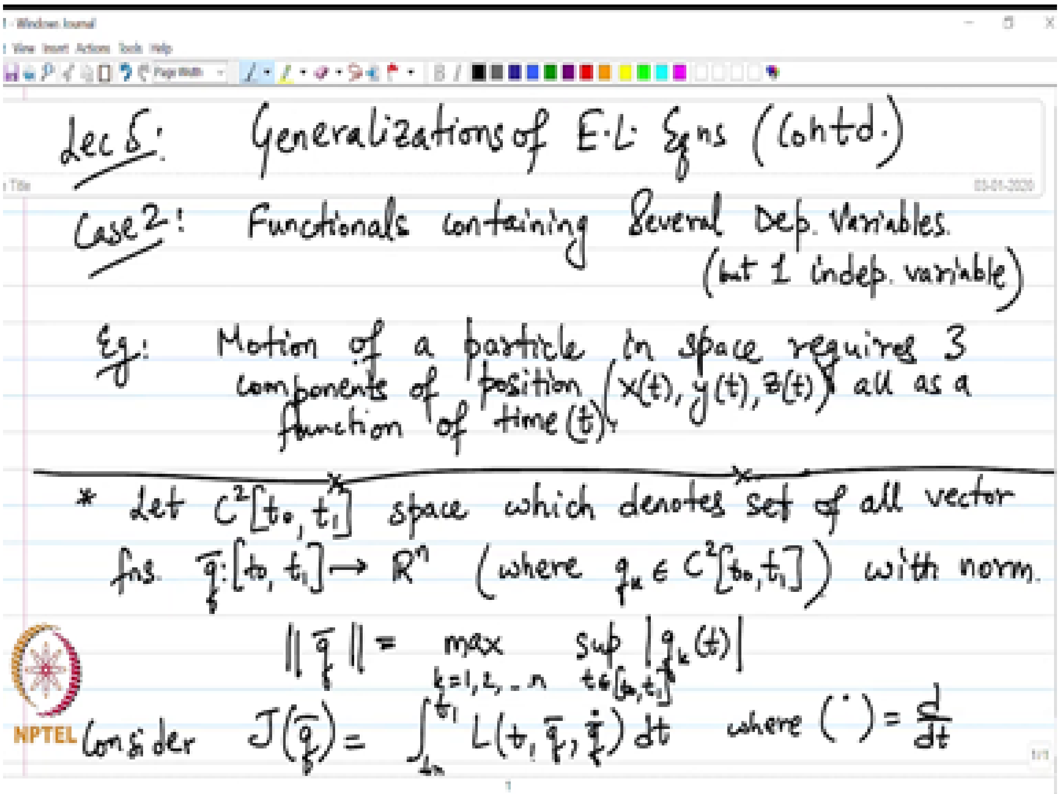

We will primarily continue our discussion that we started in the previous lecture Euler Lagrange Equations. So, this is a continued portion of the previous lecture, in this lecture we are going to do 2 major cases, namely, the generalization which involves Euler Lagrange Equation containing functions of several dependent variables but only one independent variable and the other ways, one dependent variable but several independent variables.

So, let us look at these 2 cases. We have seen generalization; the first case was functionals containing derivatives of order higher than 1. So, this is my second case, so in this case we are going to discuss how to generalize Euler Lagrange for functionals containing several independent variables, Euler Lagrange containing several independent variables but but one dependent, well, several dependent variables but one independent variable.

And then the second case will be the other way round, in this case, let me just give a brief example, so for example, let us look at the motion of a particle in 3D space will require us to track 3 independent, well, 3 components of position, let us say they are given by (x(t), y(t), z(t)) all as a function of time t.

So, this is one such case where we have 3 dependent variables x, y, z and they are all functions of the independent variables which is the time in our case and we are going to see that this also follows the standard case of the Beltrami Identity, the example that I have just mentioned. So, let us now start building up the the background to figure out the necessary condition in this case. So, first we have to describe the space where this extremal is going to be taken.

We describe $C^2[t_o, t_1]$ the set of all continuously differentiable functions up to second order within interval $[t_o, t_1]$, we describe the space which denote the set of all vector functions $\bar{q} : [t_o, t_1] \to \mathbb{R}^n$ So, the the domain of this vectors is picked up from the interval which is mainly the time and the range is all the components of this vector $\bar{q}$ which is $\mathbb{R}^n$.

So, where each of these components of $\bar{q}$ are C^2 differentiable, so where $q_k \in C^2[t_o, t_1]$ the moment we describe the space from where we are getting the vectors functions, you also have to describe the norm because that is how we are going to measure the difference when we calculate the variation of the functional, we describe the following norm

$$\|q\| = \max_{k=1,2,..,n} \sup_{t \in [t_o, t_1]} |q_k(t)|$$

Consider functional of this form $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ where $\dot{(.)} = \frac{d}{dt}$ and also that this L that I have described the integrant of this integral which is a function of this vector quantities, it is also smooth and has derivatives up to second order.

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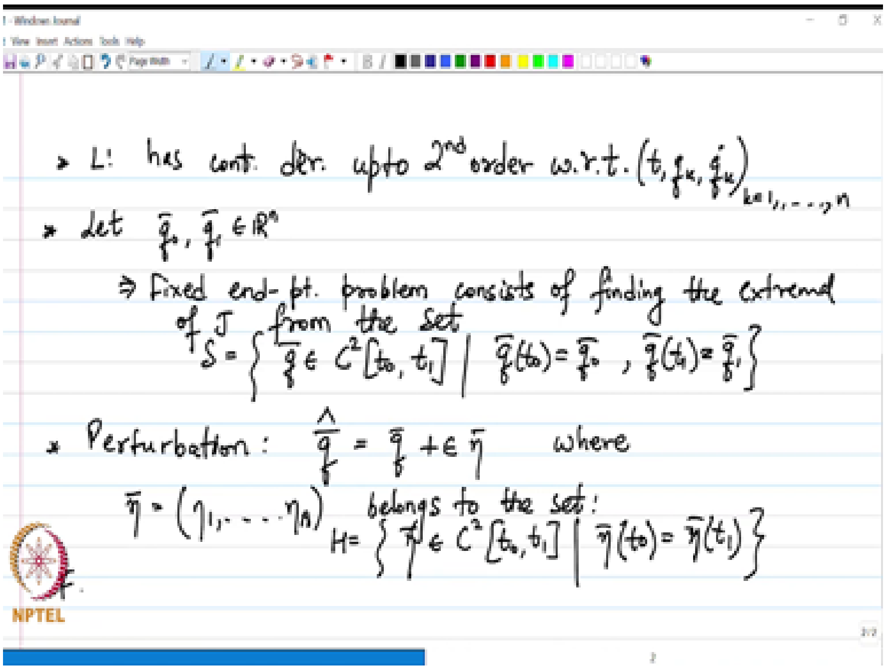

So, what I am just saying is L has continuous derivatives up to second order with respect to $(t, q_k, \dot{q_k})_{k=1,2,..,n}$ Let $q_o, q_1 \in \mathbb{R}^n$, now while we are describing our functional, notice that we also have to describe the end points because we have a fix end points case.

problem consists of finding fixed end point problem consists of finding the extremal of J from the set described by S where $S = \left\{\bar{q} \in C^2[t_o, t_1] | \bar{q}(t_o) = \bar{q}_o, \bar{q}(t_1) = \bar{q}_1\right\}$

So, then let us also further describe the perturbation of $\bar{q}$, let $\hat{\bar{q}} = \bar{q} + \epsilon\bar{\eta}$ so this is now perturbation in each of the components and where I describe my perturbation function $\bar{\eta} = (\eta_1, \eta_2, ..., \eta_k)$ which belongs which to the set $H = \left\{\bar{\eta} \in C^2[t_o, t_1] | \bar{\eta}(t_o) = \bar{\eta}(t_1)\right\}$

So, whatever we basically describe, for example, the set S, the perturbation set H in the standard derivation of Euler Lagrange, we have now extended it describe a similar sets for the vector functions. So, then now I am ready, so the moment I have described my perturbation, I am ready to expand my integrant in terms of the extremal $\bar{q}$ using Taylor Series.

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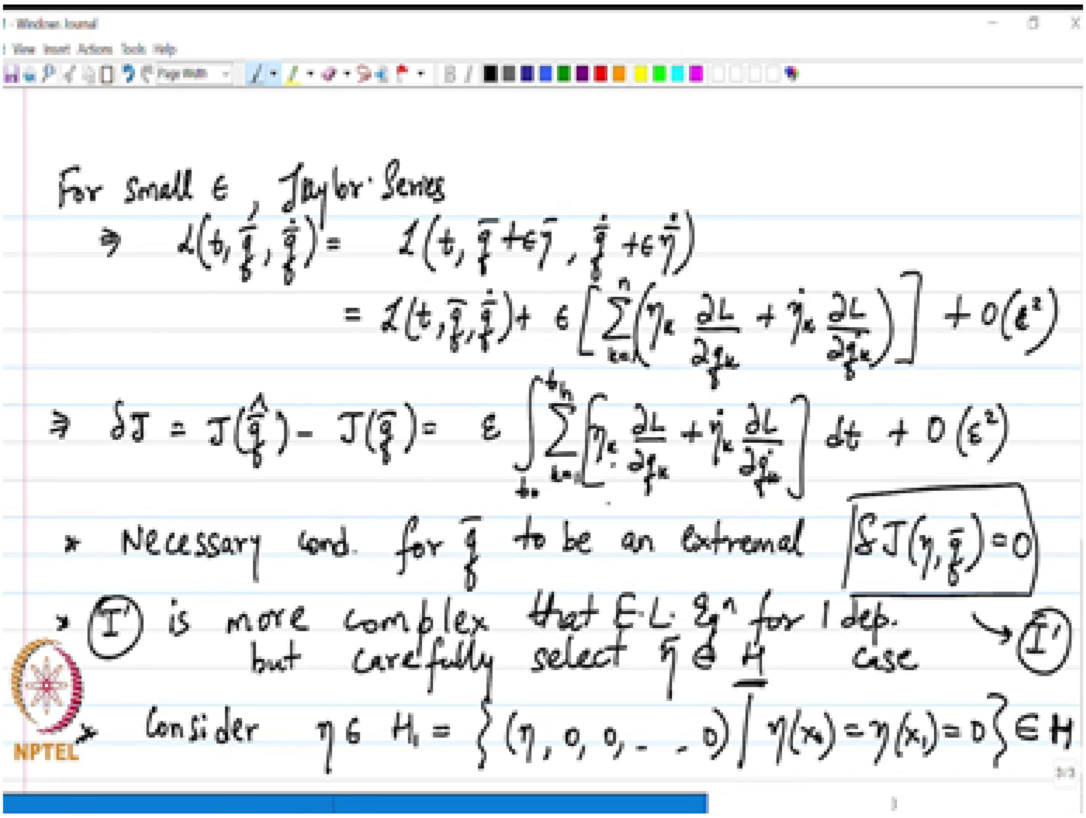

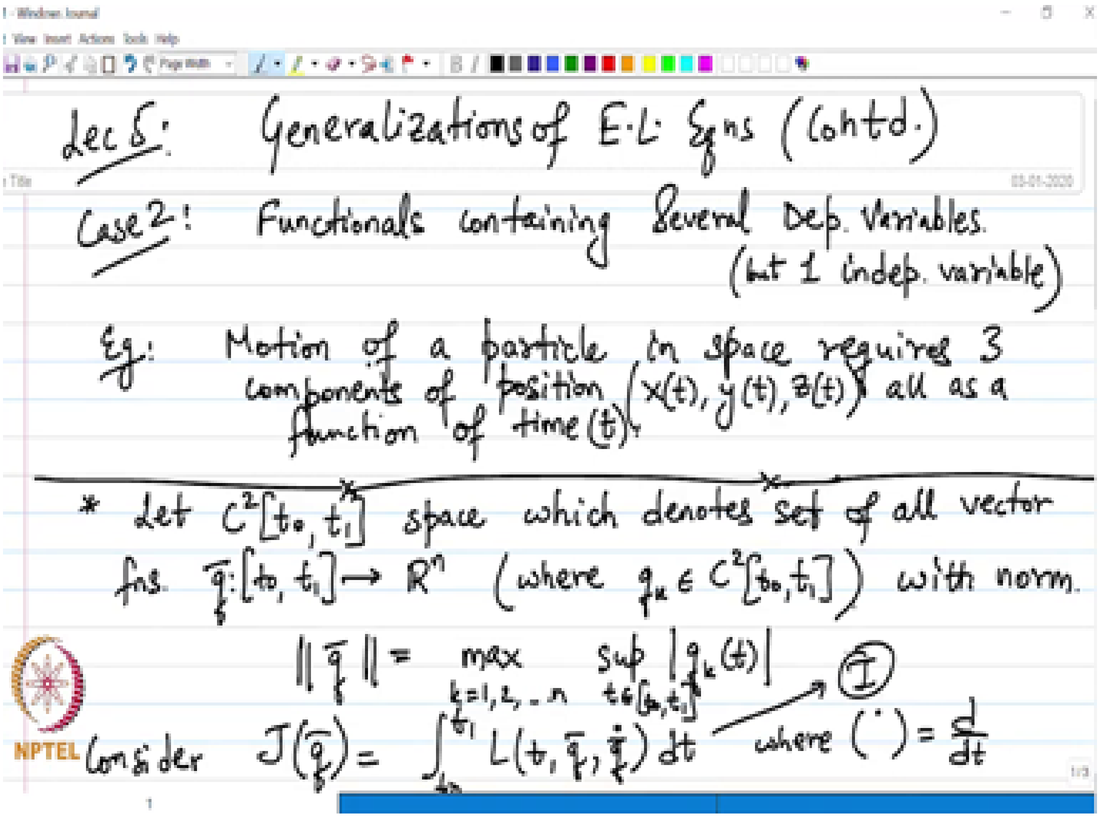

For small ϵ, the Taylor Series implies that

$$L(t,\bar{q},\dot{\hat{q}}) = L(t,\bar{q}+\epsilon\bar{\eta},\dot{\bar{q}}+\epsilon\dot{\bar{\eta}})$$

$$= L(t,\bar{q},\dot{\bar{q}}) + \epsilon\left[\sum_{K=1}^{n}\left(\eta_k\frac{\partial L}{\partial q_k} + \dot{\eta}_k\frac{\partial L}{\partial q_k}\right)\right] + O(\epsilon^2)$$

$$\Rightarrow \delta J = J(\hat{\bar{q}}) - J(\bar{q}) = \epsilon\int_{t_o}^{t_1}\sum_{K=1}^{n}\left[\eta_k\frac{\partial L}{\partial q_k} + \dot{\eta}_k\frac{\partial L}{\partial q_k}\right]dt + O(\epsilon^2)$$

Next step of finding the extremal in this particular case. we change this integral in such a way so that we can pull out these perturbations functions η_k and to do that we again further use integration by parts, so well, before that we have to understand that each of these represents, each of these η_k's they represent the components of the perturbation vector function.

We must say that the necessary condition for $\bar{q}$ to be an extremal is that we must have
$\delta J(q,\bar{q}) = 0$ $\qquad\qquad \mathbb{I}'$

So, we need to figure out the Euler Lagrange equation satisfying $\mathbb{I}'$, we also want to highlight the fact that the Euler Lagrange equation for this n component problem is slightly more complicated but we can resolve the issue as follows. So, $\mathbb{I}'$ is more complicated, more complex than E.L. equations for one dependent variable case, but we need to, but we carefully select out perturbation eta prime which is in our set of perturbations functions H.

Consider $\eta_1 \in H_1 = \{(\eta,0,0,...,0)|\eta(x_o) = \eta(x_1) = 0\} \in H$,

So, certainly this perturbation is also belongs to our perturbation set H which is this one. Now, if we use this particular perturbation, we see that our Euler Lagrange equation reduces to the 1 dependent variable case and the dependent variable being q_1.

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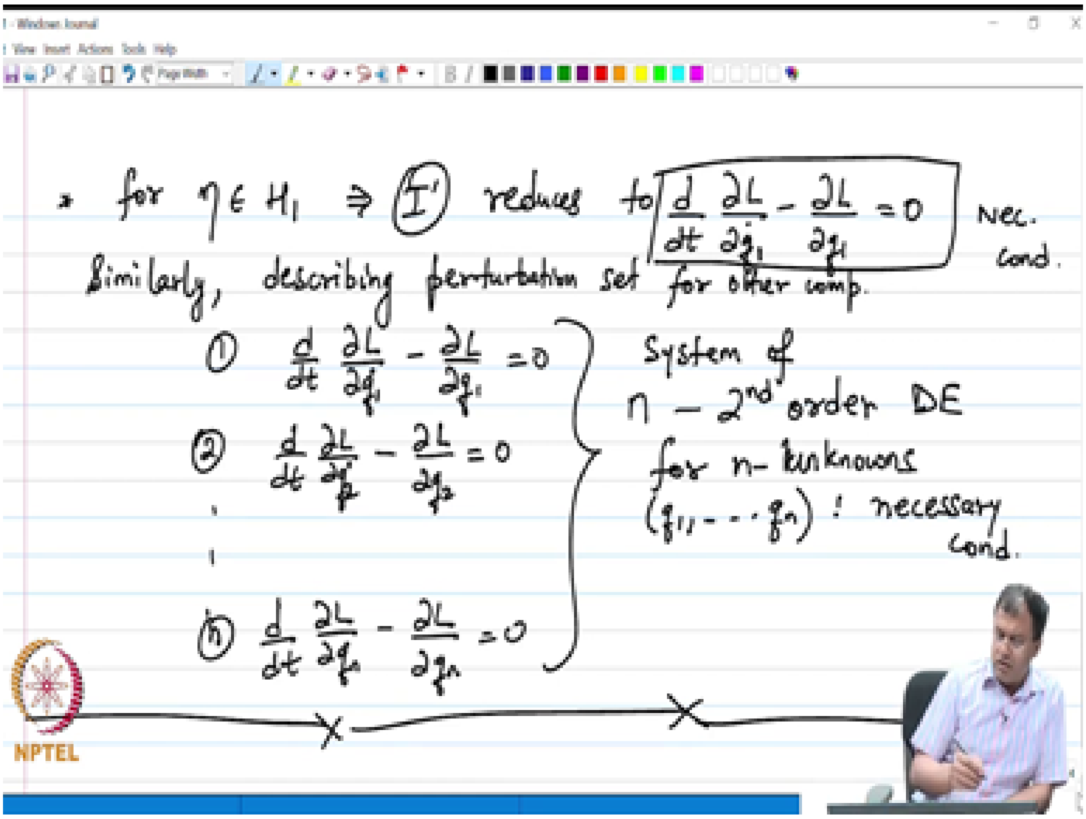

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} - \frac{\partial L}{\partial q_1} = 0$$

that is Euler Lagrange equation with respect to the first variable for the choice of perturbation that we have just chosen, so that is my necessary condition. Similarly, we can choose perturbation for other components, so similarly describing perturbation sets for other components, we are going to look, we are going to find that each of the component.

So, we can repeat this procedure like we did for the first component to the second, third, so on, and we are going to get the following set of Euler Lagrange equations, the first Euler Lagrange describes the extremal for the first component, the second Euler Lagrange equation describes for the second component, and so on so forth, the nth Euler Lagrange equation satisfies for the n^{th} component.

We are going to see that this is a system of second order differential equation for n unknowns where the unknowns are $q_1, q_2, ..q_n$which is the necessary condition for our generalized case with multiple dependent variables. So, we have found Euler Lagrange which is a system of Euler Lagrange equation. So, let me just recap so far what we have said in the form a theorem or a result.

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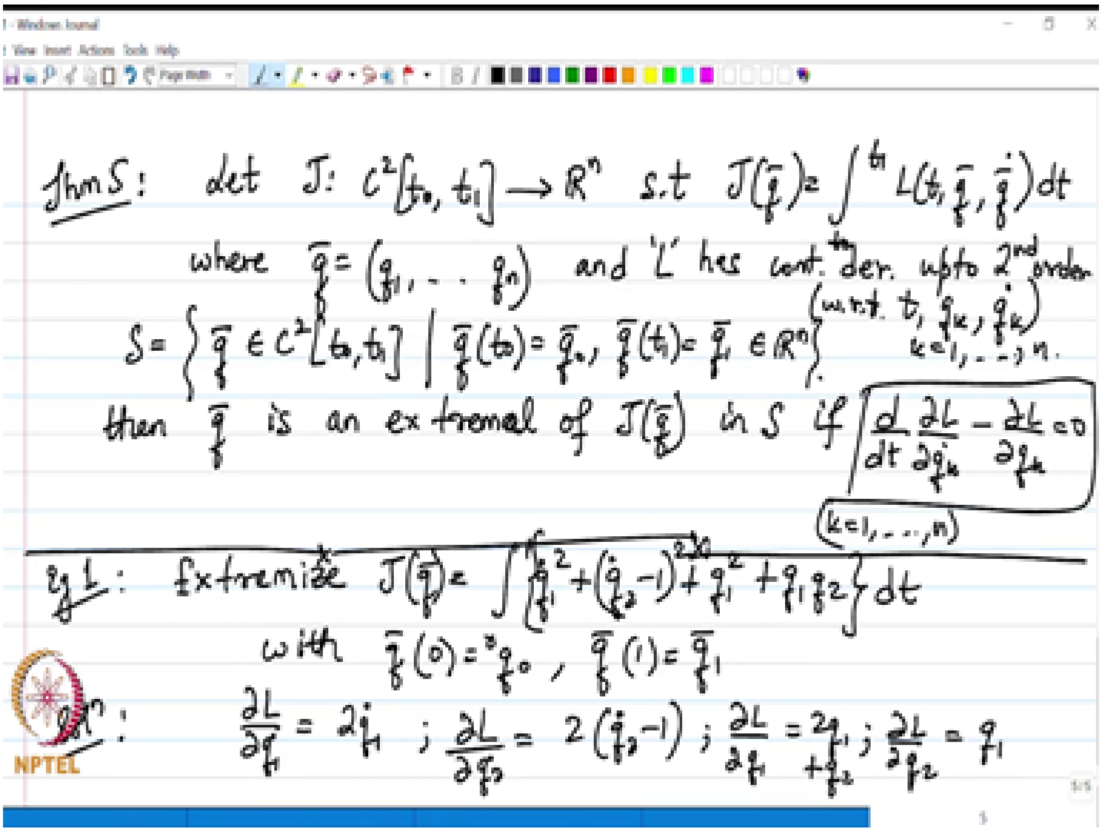

Theorem 5: Let $J : C^2[t_o, t_1] \to \mathbb{R}^n$ be a functional such that $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$
where $\bar{q} = (q_1, q_2, ...q_n)$ and the function L in side this integral has continuous derivatives up to second order with respect to t, $q_k, \dot{q_k}$, k = 1, 2,...n

and then further we describe the set S = $\left\{\bar{q} \in C^2[t_o, t_1] | \bar{q}(t_o) = \bar{q}_o, \bar{q}(t_1) = \bar{q}_1 \in \mathbb{R}^n\right\}$ and further we describe the set of perturbations as H but then the result is, then $\bar{q}$q is an extremal of J in S if $\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} - \frac{\partial L}{\partial q_k} = 0$, where k = 1, 2,...n

First example, so we need to extremize $J(\bar{q}) = \int_{t_o}^{t_1} \left\{\dot{q_1}^2 + (\dot{q_2} - 1)^2 + q_1^2 + q_1 q_2\right\} dt$, we see that in this case the functional has an integrant which has 2 dependent variables q_1 and q_2 and both depend on the independent variable t, with boundary condition $\bar{q}(0) = q_o, \bar{q}(1) = q_1$.

to find the extremals we have to write down two Euler Lagrange equation

$$\frac{\partial L}{\partial \dot{q_1}} = 2\dot{q_1}; \frac{\partial L}{\partial \dot{q_2}} = 2(\dot{q_2} - 1); \frac{\partial L}{\partial q_1} = 2q_1; \frac{\partial L}{\partial q_2} = q_1$$

So these are the different derivatives which are appearing in our Euler Lagrange and we are ready to write the system of Euler Lagrange.

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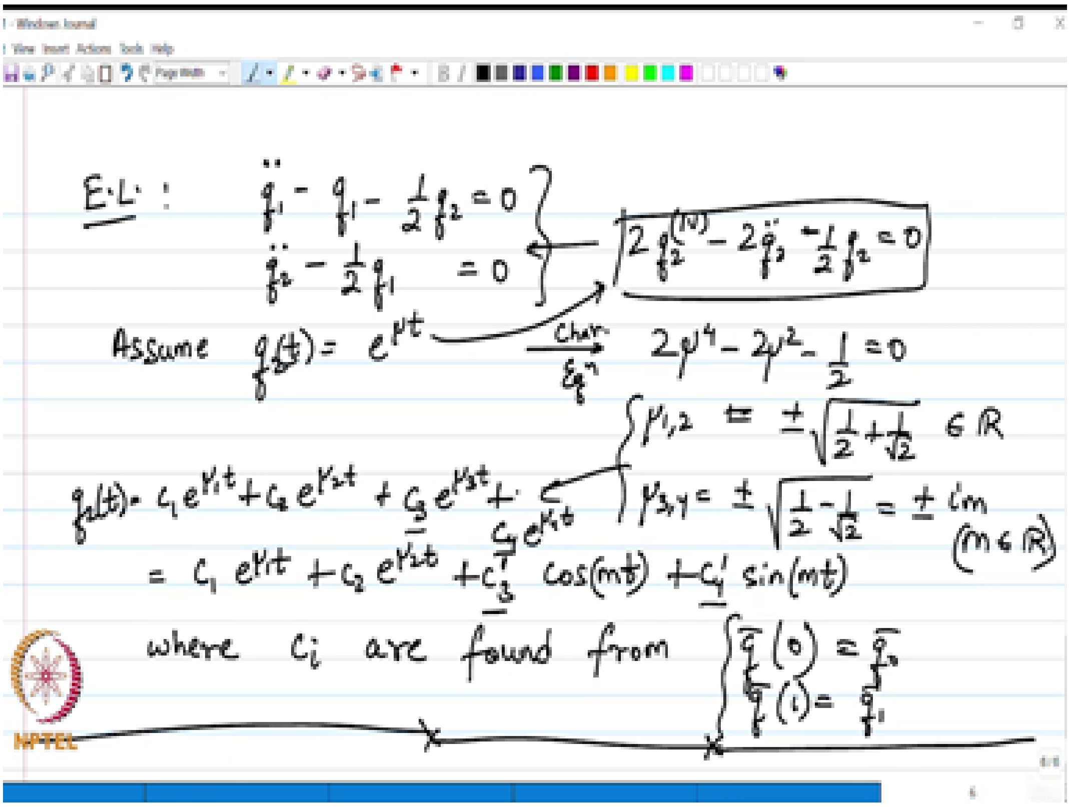

Euler Lagrange for both the variables, the first one after plugging in the values, we get the first one is $\ddot{q_1} - q_1 - \frac{1}{2}q_2 = 0$ and $\ddot{q_2} - \frac{1}{2}q_1 = 0$ So, then the next set of steps involve solving for q_1 and q_2 and standard way to do that is to eliminate one of the variables, so let us eliminate q_1 by differentiating the second equation twice with respect to t and substituting $\ddot{q_1}$ from the second equation.

what we get is an equation purely for q_2, so we get $2{q_2}^{IV} - 2\ddot{q_2} - \frac{1}{2}q_2 = 0$ So, then the next step involved is we can look at an equation, we can look at a solution for q_2 of the form, let us say $q_2(t) = e^{\mu t}$, we can plug this form here and we are going to get a characteristic equation for μ which is polynomial $2\mu^4 - 2\mu^2 - \frac{1}{2} = 0$

this is a fourth order polynomial equation, we will expect 4 solutions, the first two solutions are as follows, $\mu = \pm\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}} \in \mathbb{R}$ and the second set of solutions are given by $\mu = \pm\sqrt{\frac{1}{2} - \frac{1}{\sqrt{2}}} = im \; (m \in \mathbb{R})$

So, what I want to show here is that the second set of 2 equations are purely imaginary so it is i times some quantity which is real number, So, when we write, when we plug all these values of q, μ, we are going to get that $q_2(t) = C_1 e^{\mu_1 t} + C_2 e^{\mu_2 t} + C_3 e^{\mu_3 t}$ we get the following linear combination, and we see that after plugging in, after simplifying we expect that since the third and the fourth solution, they are occurring in complex conjugates, we can always write the solution in the form of sin and cos.

So, I get that this is also equal to C_3, let me say that we will have a different constant altogether, so $C_3' \cos{(mt)} + C_4' \sin{(mt)}$, now we have got a family of extremals, I can find the $C_i's$ are found from the boundary condition $\bar{q}(0) = q_o$ and $\bar{q}(1) = q_1$ and each of them are set of two equations so we have a total of 4 equations for 4 unknowns. And that completes the discussion of this example.

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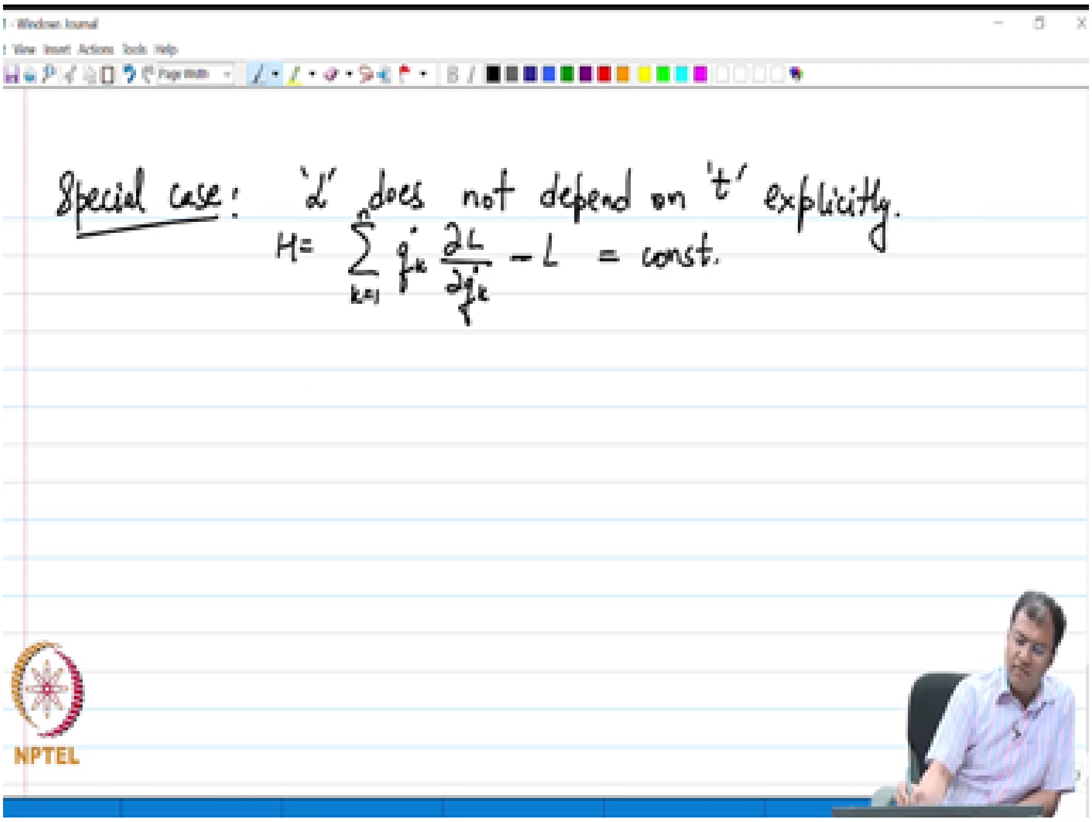

Another example, let us look at a special case, where L does not explicitly depend on t, that is the case where we should have a Beltrami Identity instead of the full Euler Lagrange equation and that identity must be the reduced order Euler Lagrange equation. So, in the special case L, the integrant in the functional does not depend on t explicitly.

So, I have that $H = \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = Constant$, we see that when L does not contain explicitly the t, the independent variable then in this case the Beltrami Identity is of this following form where we have the summation over is variables $q_k'^{}s$ so this is my Beltrami Identity.