Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 12 – Existence and Uniqueness of solutions - Part 3

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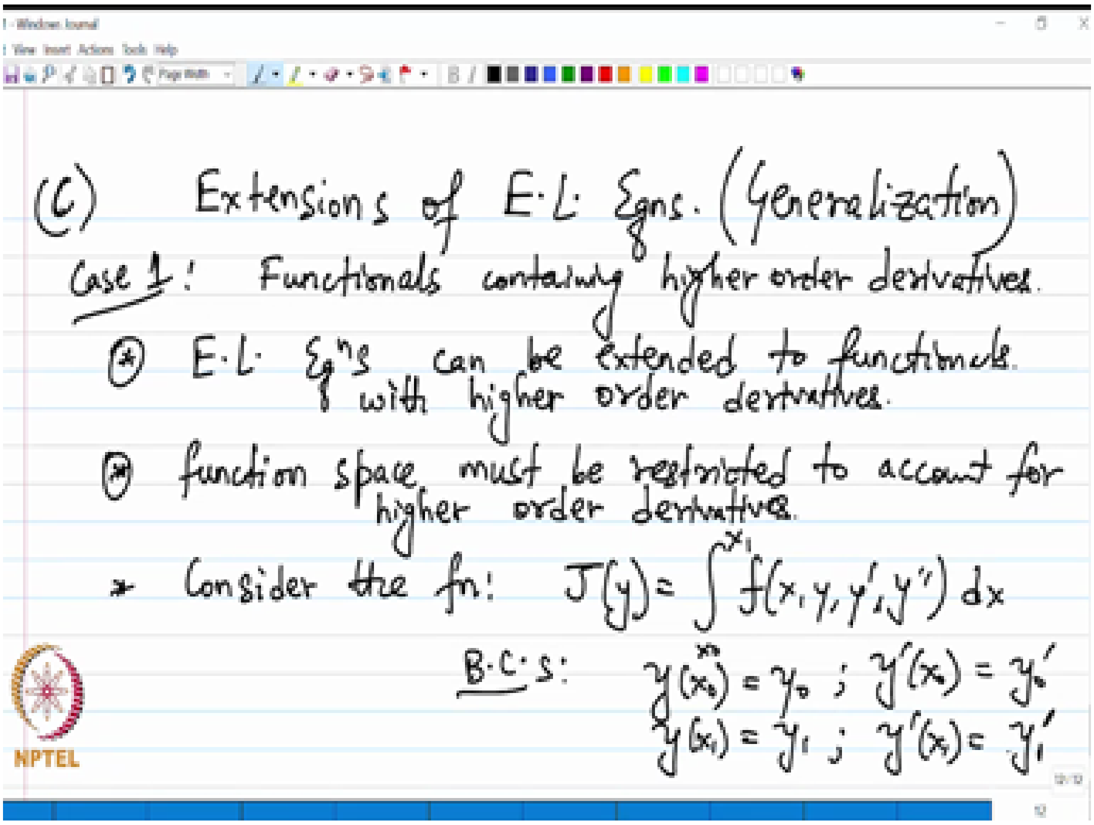

Let us now move ahead with another topic and another point of discussion namely, the generalization of Euler-Lagrange equation. So, I call this as case **C**. So, what I am going to discuss in this third case is extensions of Euler-Lagrange equations or in short, I would like to generalize my Euler-Lagrange equations for functionals containing integrants having higher derivatives and several other generalizations which we will see one by one.

So, under this let us look at the first type of generalization, let me call this as Case 1, in this case we look at functionals containing higher order derivatives. So, in this case points that are to be noted that we can certainly extend our result of the Euler-Lagrange equation to functions, which are the integrands of the functionals and which contains higher order derivatives, let us say derivatives like $y^{''}, y^{'''}$ and so on so forth.

Euler-Lagrange equations can be extended, can be extended to functionals with higher order derivatives and further the another issue that we have to consider is the moment we are considering higher order derivatives, our function space will be restricted.

So, instead of searching for our extremal in $C^2[x_o, x_1]$ we have to look at C^3 or C^4 and so on so forth. So, our function from which the extremal comes out, the function space must be restricted to account for higher orderderivatives.

let us now consider the functional $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'}, y^{''})$ **I**
We have now a function containing derivatives up to second order. So, the moment we introduce extremals having continuous derivatives up to second order we have to provide extra set of boundary conditions.

My fixed point boundary conditions $y(x_o) = y_o$; $y^{'}(x_o) = y_o^{'}$, $y(x_1) = y_1$; $y^{'}(x_1) = y_1^{'}$ which means now earlier we were assuming that the extremal y had continuous partial derivatives up to second order. Now, we are going to assume that the extremals y will have continuous partial derivatives up to third order or in other words, y comes from the space C^4. So, let me show how is it possible.

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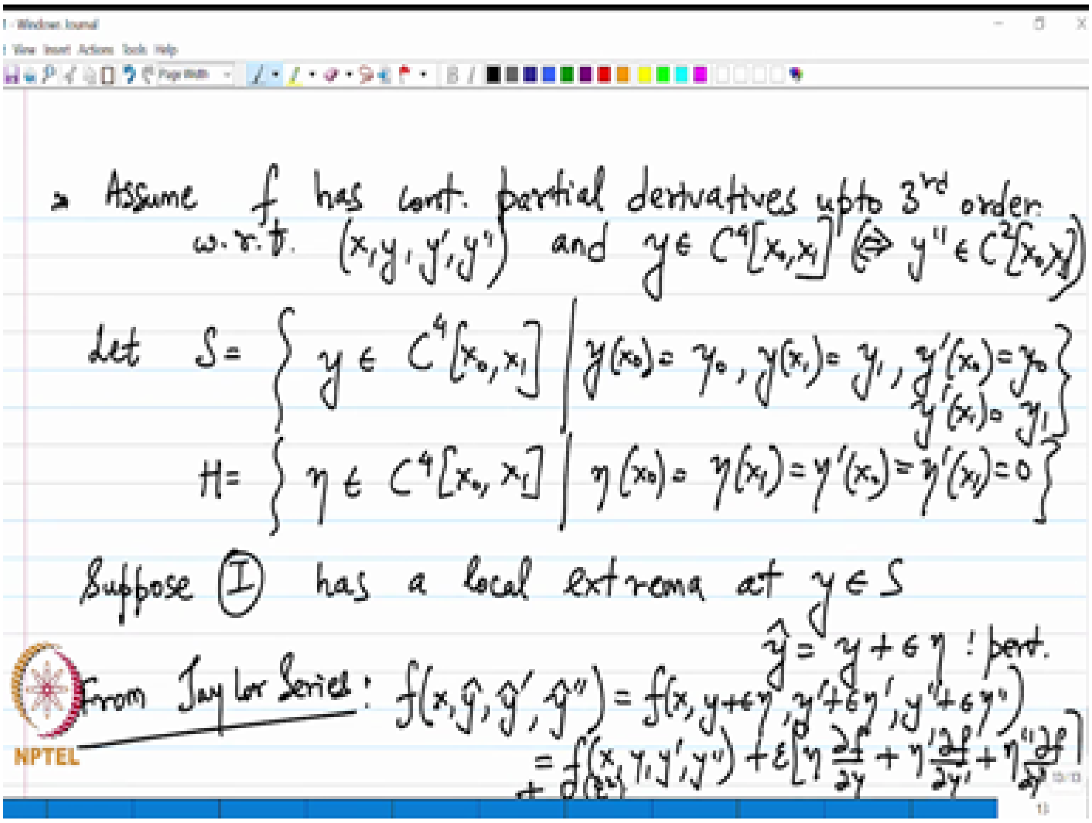

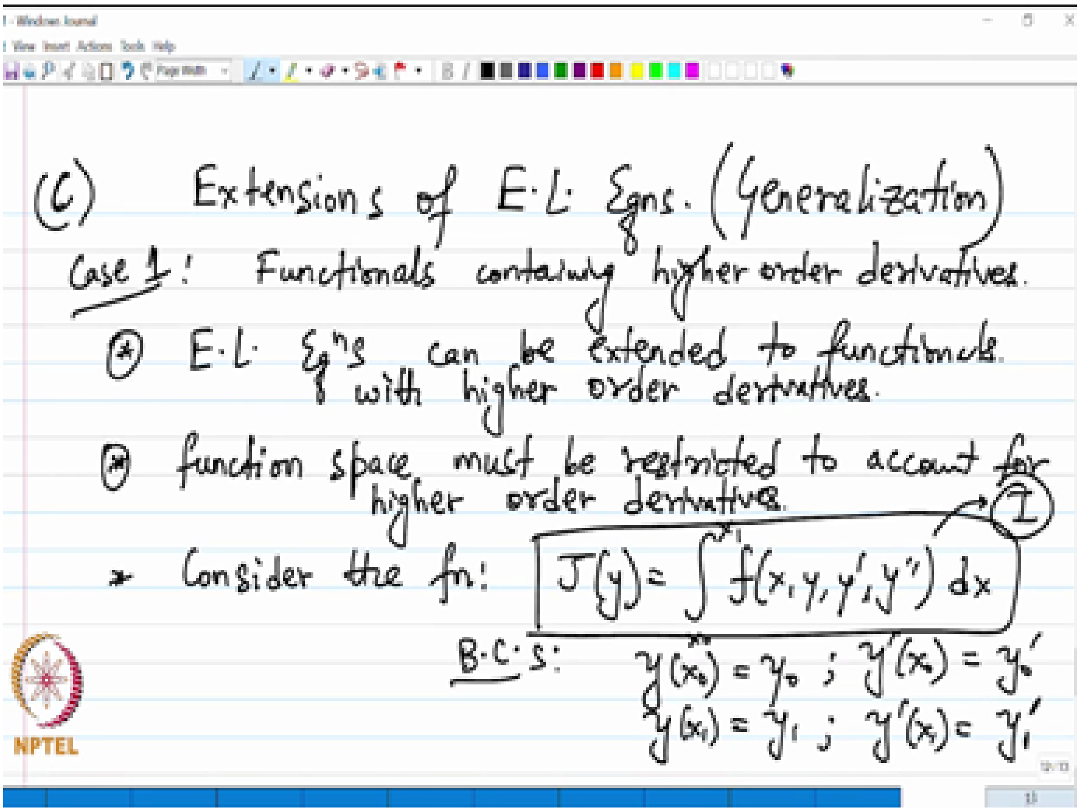

So, what we have is the following: Assume f has continuous partial derivative up to third order with respect to the variables x, y, $y^{'}$ and $y^{''}$ and this means that y must come from $C^4[x_o, x_1]$ why C_4? Because this is equivalent to saying that y double derivative must come from $C^2[x_o, x_1]$. Because we have derivatives of y up to second order and for our Euler-Lagrange equation we need the variable with the highest derivative to be at least C^2 or the variable that we have y must be C^4.

So, then we have to rewrite our function space. So, now my S becomes

$$S = \Big\{ y \in C^4[x_o, x_1] | y(x_o) = y_o, y(x_1) = y_1, y^{'}(x_o) = y_o, y^{'}(x_1) = y_1 \Big\}$$

$$H = \Big\{ \eta \in C^4[x_o, x_1] | \eta(x_o) = \eta_o, \eta(x_1) = \eta_1, \eta^{'}(x_o) = \eta_o, \eta^{'}(x_1) = \eta_1 \Big\}$$

So, the perturbation set is such that the values as well as the values of the first derivative at the boundary points vanish. So, with this we are ready to describe our Euler-Lagrange equation or the necessary condition for the existence of the extremal. So, suppose **I** has a local extrema at the function $y \in S$, and let us say that the perturbation to this extrema is $\hat{y} = y + \epsilon\eta$.

From Taylor series expansion, I can rewrite my integrand in **I**,

$$f(x, \hat{y}, \hat{y}^{'}, \hat{y}^{''}) = f(x, y + \epsilon\eta, y^{'} + \epsilon\eta^{'}, y^{''} + \epsilon\eta^{''})$$

$$= f(x, y, y^{'}, y^{''}) + \epsilon \left[\eta \frac{\partial f}{\partial y} + \eta^{'} \frac{\partial f}{\partial y^{'}} + \eta^{''} \frac{\partial f}{\partial y^{''}} \right] + O(\epsilon^2)$$

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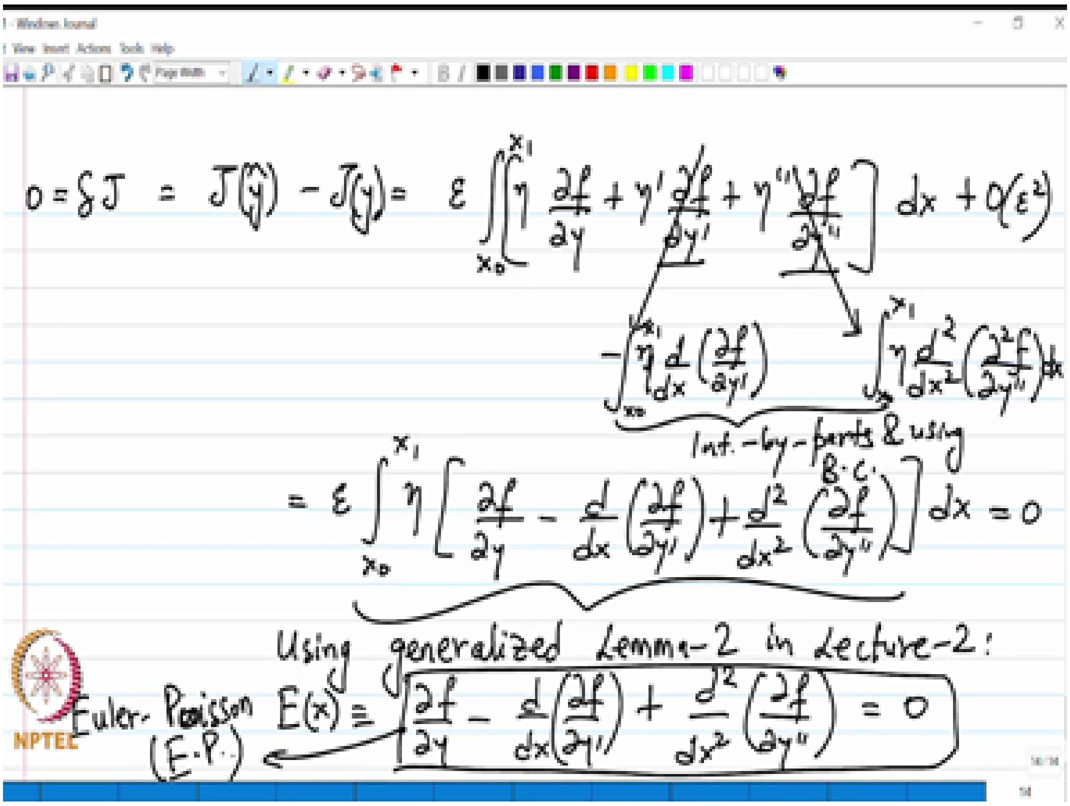

$$0 = \delta J = J(\hat{y}) - J(y) = \epsilon \int_{x_o}^{x_1} \left[\eta \frac{\partial f}{\partial y} + \eta^{'} \frac{\partial f}{\partial y'} + \eta^{''} \frac{\partial f}{\partial y''} \right] + O(\epsilon^2)$$

So, then we do, redo the exercise that we did in the standard Euler-Lagrange case, that is we are going to change the second and the third term using integration by parts and then we apply the boundary condition

$$\eta^{'} \frac{\partial f}{\partial y'} = - \int_{x_o}^{x_1} \eta \frac{d}{dx} \frac{\partial f}{\partial y'} dx$$

$$\eta^{''} \frac{\partial f}{\partial y''} = \int_{x_o}^{x_1} \eta \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} dx$$

$$= \epsilon \int_{x_o}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} \right] dx = 0$$

we see that the extremal, again we use a similar variation of lemma 2 that we showed in our lecture 2 to come to the fact that, we must have that the extremal is 0 provided this variation is 0 or which means that this variation must be 0 for epsilon sufficiently small and from here we come to the fact that using, I must say that using the generalized lemma 2 discussed in our lecture number 2, we arrive at the fact that

$$E(x) \equiv \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0$$

So, this is the extension of the Euler-Lagrange equation for functions containing derivatives up to second order, we call this extended Euler-Lagrange as Euler-Poisson equation, I denote it in shorthand notation as EP equation. So, the extended Euler-Lagrange equations for functions containing higher order derivatives are also known as Euler-Poisson equations. So let us look at an example in this situation.

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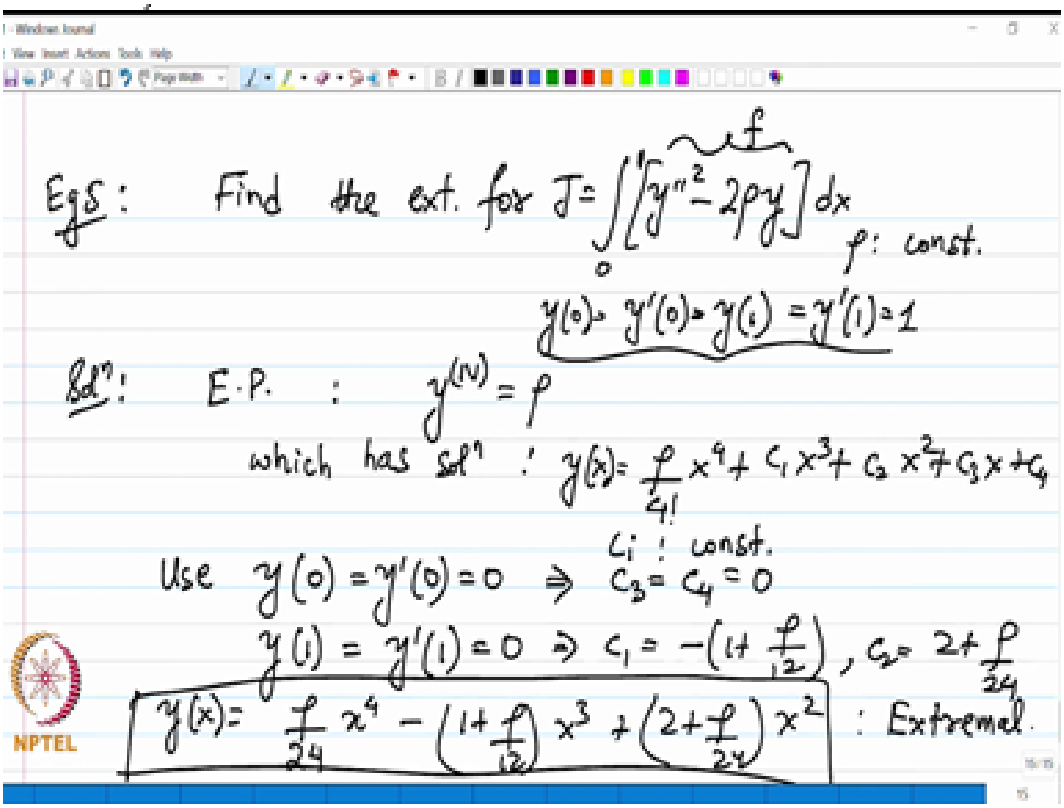

let us look at this particular example. find the extremal for $J(y) = \int_0^1 [y''^2 - 2\rho y]dx$, where ρ is a constant and $y(0) = y'(0) = y(1) = y'(1) = 1$

we have to find the extremal for this functional. Notice that this functional contains derivatives up to second order and my Euler-Lagrange equation or my generalized Euler-Lagrange or Euler-Poisson equation, EP equation, it reduces to after plugging in this expression for f in Euler Poisson, you see that my equation reduces to the following fourth order ODE or differential ordinary differential equation, which has the solution of the form

$$y(x) = \frac{\rho}{4!}x^4 + C_1x^3 + C_2x^2 + C_1x + C_4$$

where C_i are constant So, using all these we have four constants, and we have four boundary conditions, we can very conveniently eliminate all these unknowns of the problem. If we use the fact that the first of the two boundary conditions $y(0) = y'(0) = o \Rightarrow C_3 = C_4 = 0$

Further, if we use the fact that $y(1) = y'(1) = 0 \Rightarrow C_1 = -\left(1 + \frac{\rho}{12}\right)$ and $C_2 = 2 + \frac{\rho}{24}$ It is all about a matter of algebra to solve all these constraints and finally, expression for the extremal is the following polynomial:

$$y(x) = \frac{\rho}{24}x^4 - \left(1 + \frac{\rho}{12}\right)x^3 + \left(2 + \frac{\rho}{24}\right)x^2$$

this is my extremal to this fourth order functional. So, that completes the solution to this example. So, let us look at some specific cases in this generalized scenario.

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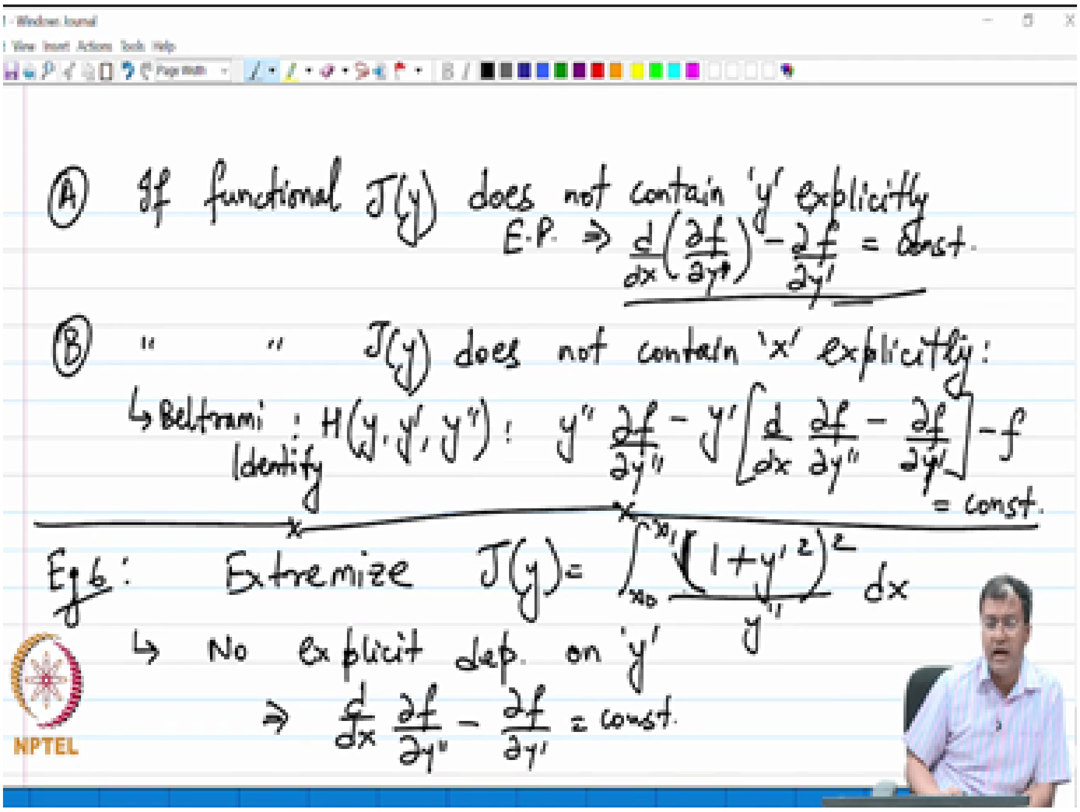

So, there are two specific cases I want to highlight, let me call he cases as A and B. The first case is if my functional J(y) does not contain y explicitly then my Euler-Poisson reduces to the following equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right) - \frac{\partial f}{\partial y'}$ = constant. It comes via the direct removing certain unnecessary terms, where we have involved derivatives of y and then integrating once to get, to come to this point.

So, instead of solving the fourth order ODE, we are solving a second order ODE, or third order ODE in this case, The second special case is, if my functional J(y) does not contain x explicitly, then my Euler-Poisson reduces to the generalized Beltrami identity. So, I have this is the case where we have the generalized Beltrami identity to be satisfied, which is basically the following function

$$H(y, y', y'') = y'' \frac{\partial f}{\partial y''} - y' \left[\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right] - f = Constant$$

So, that is my generalized Beltrami identity that we instead solve in this special case, again this is a reduced order Euler-Poisson equation,

let us look at an another example in this discussion, let us extremize this functional

$$J(y) = \int_{x_o}^{x_1} \frac{(1 + y'^2)^2}{y''} dx$$

we see that we do not have an explicit dependence on y in this particular integrand. So, we, this particular integrand falls under the case scenario A.

So, what we can do is that my Euler-Lagrange or my Euler-Poisson equation reduces to $\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}$ = constant, So, once we plug in all the values of the expression we should be able to solve.

However, there is a further simplification that we can perform. Notice that this integrand is also independent of any x. So, there is no explicit x dependence on this integrant, which means that we can also use the Beltrami identity. So, what we said is the following:

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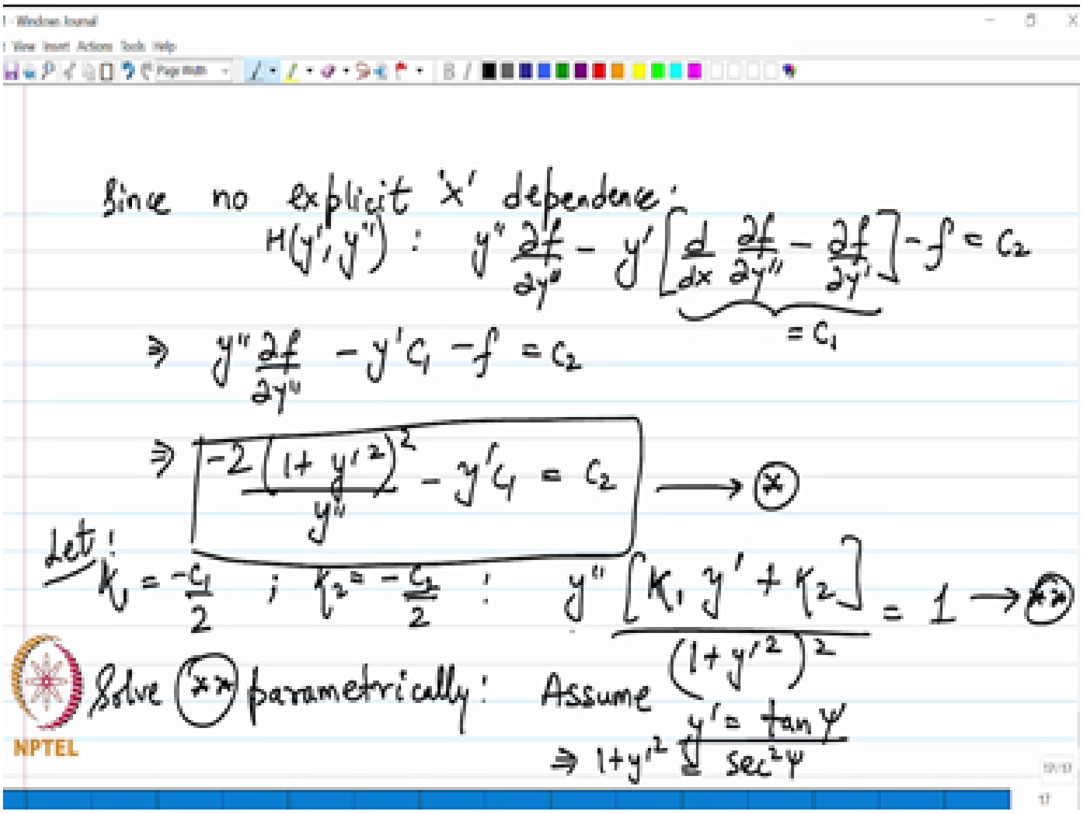

$$H(y^{'}, y^{''}) = y^{''}\frac{\partial f}{\partial y^{''}} - y^{'}\left[\frac{d}{dx}\frac{\partial f}{\partial y^{''}} - \frac{\partial f}{\partial y^{'}}\right] - f = C_2$$

$$\Rightarrow y^{''}\frac{\partial f}{\partial y^{''}} - y^{'}C_1 - f = C_2$$

$$\Rightarrow \frac{-2(1+y^{'2})^2}{y^{''}} - y^{'}C_1 = C_2 \qquad *$$

So that is a simplified expression. So then, let $\kappa_1 = -\frac{C_1}{2}$ and $\kappa_2 = -\frac{C_2}{2}$

we are again going to solve it parametrically or we are going to find y as a function of a parameter and x as a function of the same common parameter and that will be our final solution, just first rewrite this expression in terms of these new constants

$$\frac{y^{''}[\kappa_1 y^{'} + \kappa_2]}{(1+y^{'2})^2} = 1 \qquad **$$

we solve double star parametrically, we solve double star parametrically from now on by assuming by again assuming that $y^{'} = \tan\psi \Rightarrow 1 + y^{'2} = \sec^2\psi$
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$$\Rightarrow y'' = \sec^2\psi \; \psi' \qquad \left(\psi' \rightarrow \frac{d\psi}{dx}\right)$$

$$(**) : \left[\kappa_1 \cos\psi \sin\psi + \kappa_2 \cos^2\psi\right] \psi' = 1 \longrightarrow (***)$$

Integrate $(***)$:

$$x = \kappa_3 + 2\kappa_2 \psi + \kappa_2 \sin(2\psi) - \kappa_1 \cos(2\psi) \longrightarrow (4*)$$

Note $y' = \tan\psi \longrightarrow dy = \tan\psi \, dx$

Using $(4*)$ & Integrate w.r.t (ψ):

$$y = \kappa_4 + 2\kappa_1 \psi - \kappa_2 \cos(2\psi) + \kappa_1 \sin(2\psi) \longrightarrow (5*)$$

$(4*, 5*)$: Parametric Sol^n $(x(\psi), y(\psi))$

Since no explicit 'x' dependence:

$$H(y', y'') : \; y'' \frac{\partial f}{\partial y''} - y' \underbrace{\left[\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right]}_{= C_1} - f = C_2$$

$$\Rightarrow y'' \frac{\partial f}{\partial y''} - y' C_1 - f = C_2$$

$$\Rightarrow \boxed{-2\frac{(1+y'^2)^2}{y''} - y' C_1 = C_2} \longrightarrow (*)$$

Let: $\kappa_1 = \frac{-C_1}{2}$; $\kappa_2 = -\frac{C_2}{2}$:

$$y'' \frac{\left[\kappa_1 y' + \kappa_2\right]}{(1+y'^2)^2} = 1 \longrightarrow (**)$$

Solve $(**)$ parametrically: Assume $y' = \tan\psi$

$$\Rightarrow 1 + y'^2 = \sec^2\psi$$

$y^{''} = \sec^2 \psi \psi^{'}$

$$\left[\kappa_1 \cos\psi \sin\psi + \cos^2\psi\right]\psi^{'} = 1 \qquad ***$$

Integrate *** we get

$$x = \kappa_3 + 2\kappa_2\psi + \kappa_2 \sin 2\psi - \kappa_1 \cos 2\psi \qquad \mathbf{4*}$$

Note: $y^{'} = \tan\psi dx \Rightarrow dy = \tan\psi dx$, Using **4*** and Integrate w.r.t to ψ we get

$$y = \kappa_4 + 2\kappa_1\psi - \kappa_2 \cos 2\psi + \kappa_1 \sin 2\psi \qquad \mathbf{5*}$$

So, my **4*** and **5***are parametric solution given by $(x(\psi), y(\psi))$ So, we are going to end the discussion in this lecture by mentioning one important result, namely that the Euler-Poisson equation can be further generalized for functionals containing integrands of higher and higher derivative. So, what I meant to say is the following:

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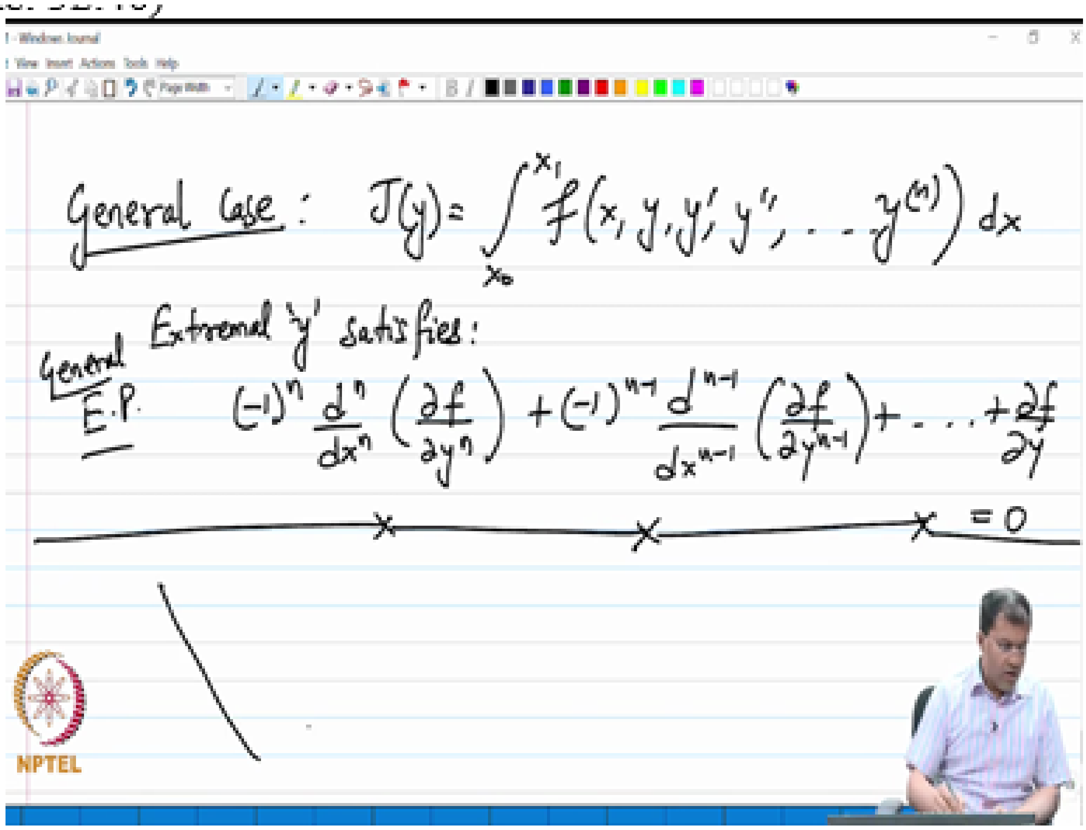

Let us look at this most general case $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'}, y^{''}, ...y^{(n)})dx$ the function integrand contains the derivatives of y up to the order n, the extremal y satisfies the Euler-Poisson equation which is of this following form.

Notice that this conveniently reduces to the Euler-Lagrange as well as the Euler-Poisson equation for functionals containing derivatives up to second order. So, this is the general version of Euler-Poisson.

$$(-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^n}\right) + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial f}{\partial y^{n-1}}\right) + ... + \frac{\partial f}{\partial y} = 0$$

The last quantity that we will have is the derivative of partial of f with respect to y and this is set equal to 0. So, this is my generalized Euler-Poisson for functions containing derivatives of any order. let us end the session by mentioning that we have several other ways to generalize the Euler-Poisson namely, how about looking at the case where we have multiple dependent variables or we have multiple independent variables, we will see that those sort of equations frequently arise in continuum mechanics, especially in planetary motion, as well as standard Newtonian mechanics. Thank you for listening for this lecture.