Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 11 – Existence and Uniqueness of solutions - Part 2

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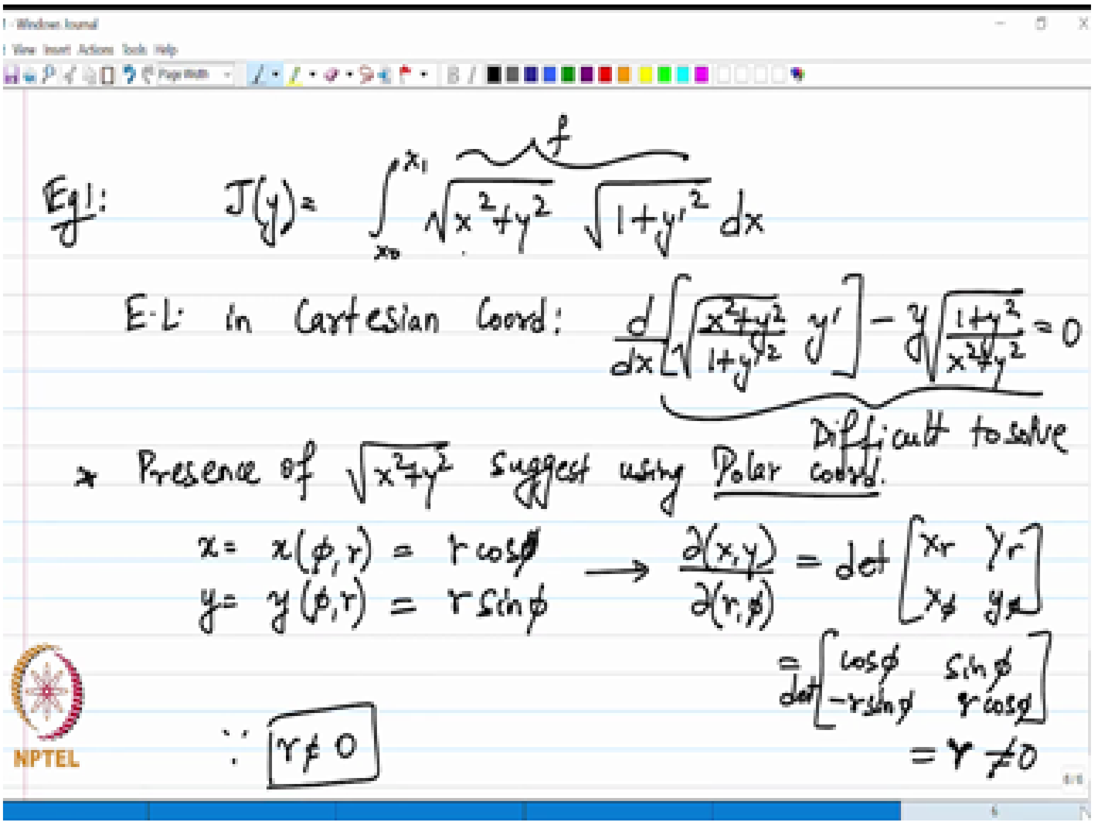

the first example that I have in this lecture course is,

$$J(y) = \int_{x_o}^{x_1} \sqrt{x^2 + y^2}\sqrt{1 + {y'}^2}dx$$

notice that J depends on all the three variables x, y and $y^{'}$. Which means if we were to solve for the extremal in Cartesian coordinates, look at the complexity of this Euler-Lagrange equation.

$$\frac{d}{dx}\left[\sqrt{\frac{x^2 + y^2}{1 + {y'}^2}}y^{'}\right] - y\sqrt{\frac{1 + y^2}{x^2 + y^2}} = 0$$

notice that this Euler-Lagrange equation is a mess, this is extremely difficult to solve.

You can still attempt but there is an easier way out. The easier way out is by the observation that we have the formation of this expression $\sqrt{x^2 + y^2}$ So, if you recall that in polar coordinate my radius variable r is nothing but $\sqrt{x^2 + y^2}$ suggests that we should possibly change this functional in polar coordinates r, θ or r, ϕ.

x $= x(\phi, r) = r\cos\phi$ and $y = y(\phi, r) = r\sin\phi$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} x_r & y_r \\ x_\phi & y_\phi \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{vmatrix} = r \neq 0$$

r is not 0. Otherwise the transformation is bogus. It is not a non-singular transformation. So, this clearly shows that the determinant is non-singular and hence we are ready to transform our original functional in Cartesian frame into the polar frame functional.

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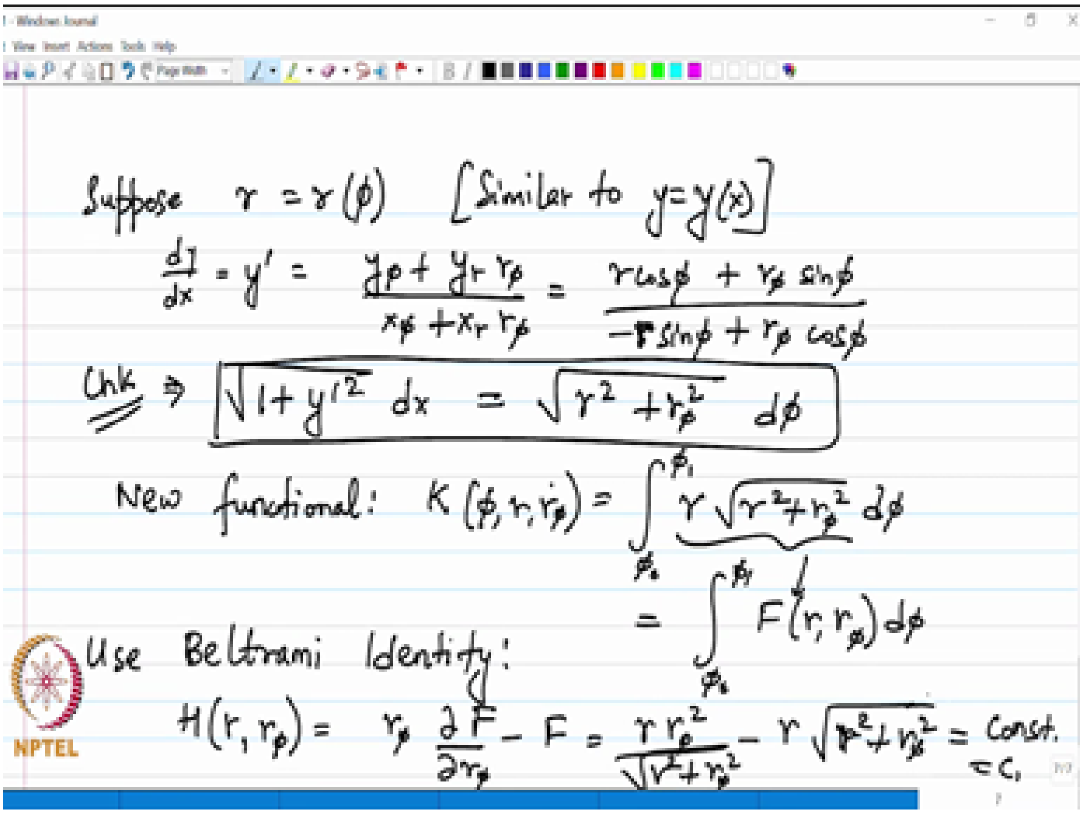

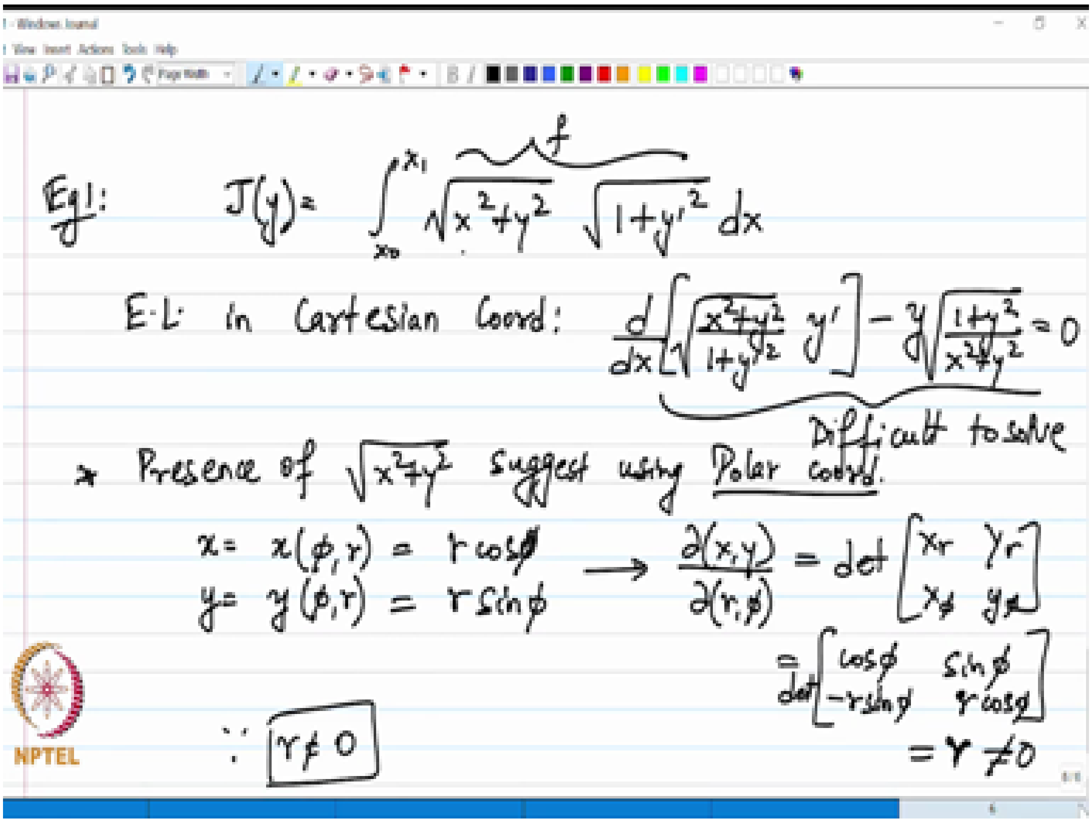

So, then let us further assume like we assume in Cartesian frame, suppose r = $r(\phi)$ so r is dependent on ϕ. Similar to y dependent on x that we had in the Cartesian frame.

$$\frac{dy}{dx} = \frac{y_\phi + y_r r_\phi}{x_\phi + x_r r_\phi} = \frac{r\cos\phi + r_\phi \sin\phi}{-r\sin\phi + r_\phi\cos\phi}$$

$$\Rightarrow \sqrt{1+{y'}^2}dr = \sqrt{r^2 + {r_\phi}^2}d\phi$$

$$New functional \qquad K(\phi, r, r_\phi) = \int_{\phi_o}^{\phi_1} r\sqrt{r^2 + {r_\phi}^2}d\phi = \int_{\phi_o}^{\phi_1} F(r, r_\phi)d\phi$$

So we can peacefully use the Beltrami identity because the explicit dependence on ϕ is missing in this integrand. So, we are going to use Beltrami identity and reduce, thereby reducing our Euler-Lagrange equation to first order ordinary differential equation. So, Beltrami identity

$$H(r, r_\phi) = r_\phi \frac{\partial F}{\partial r_\phi} - F = \frac{r{r_\phi}^2}{\sqrt{r^2 + {r_\phi}^2}} - r\sqrt{r^2 + {r_\phi}^2} = Constant = C_1$$

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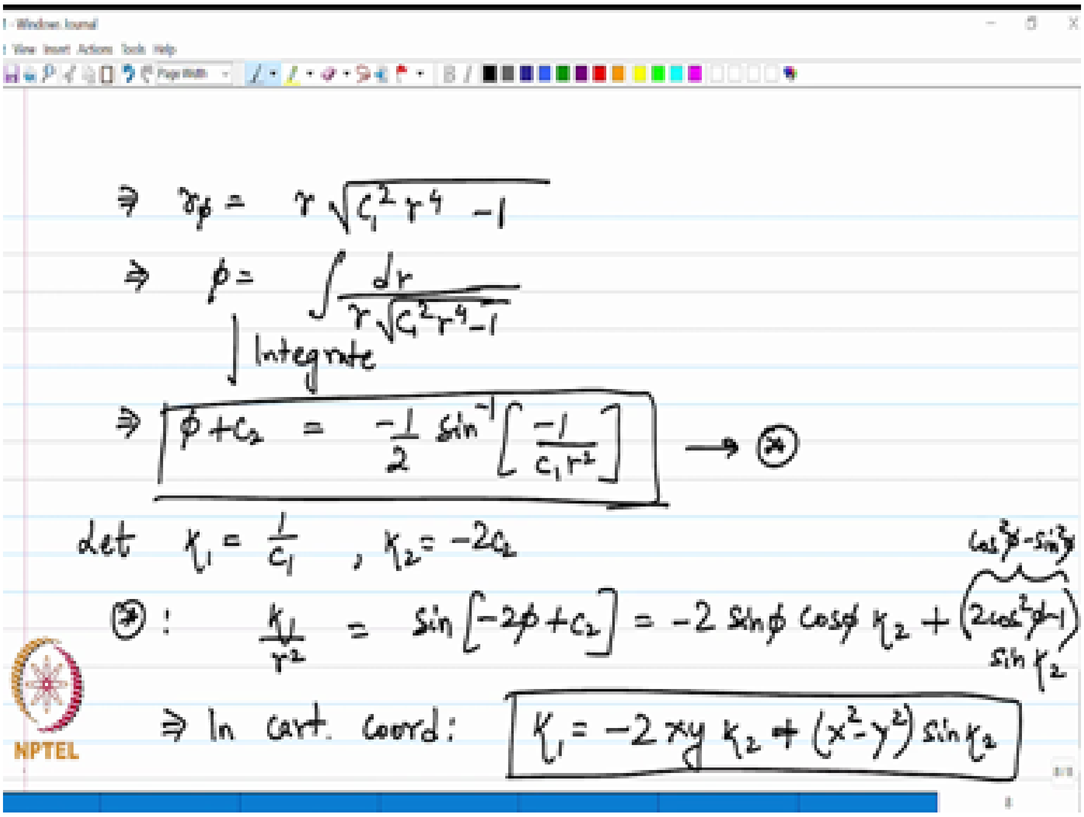

$$\Rightarrow r_\phi = r\sqrt{{C_1}^2 r^4 - 1}$$

$$\Rightarrow \phi = \int \frac{dr}{r\sqrt{{C_1}^2 r^4 - 1}}$$

$$\Rightarrow \phi + C_2 = -\frac{1}{2}\sin^{-1}\left[\frac{-1}{C_1 r^2}\right] \qquad *$$

So, the extremal is in this form although it is not clear what is the form of this extremal directly from this expression. So, we do a little bit of simplification, we assume another set of constants. So, let new constraints $\kappa_1 = \frac{1}{C_1}$ and $\kappa_2 = -2C_2$

$$* : \qquad \frac{\kappa_1}{r^2} = \sin\left[-2\phi + C_2\right] = -2\sin\phi\cos\phi\kappa_2 + (2\cos^2\phi - 1)\sin\kappa_2$$

In Cartesian coordinate $\kappa = -2xy\kappa_2 + (x^2 - y^2)\sin\kappa_2$

So, that is the extremal that we have found in Cartesian frame. So, notice that we have now found the extremal in the Cartesian frame without ever solving the Euler-Lagrange equation in the Cartesian frame. So, it is via the polar frame that we are able to systematically integrate our equation for the extremal. So, doing a necessary, performing a necessary coordinate transformation helps to simplify the problem, and the choice must be judicious.

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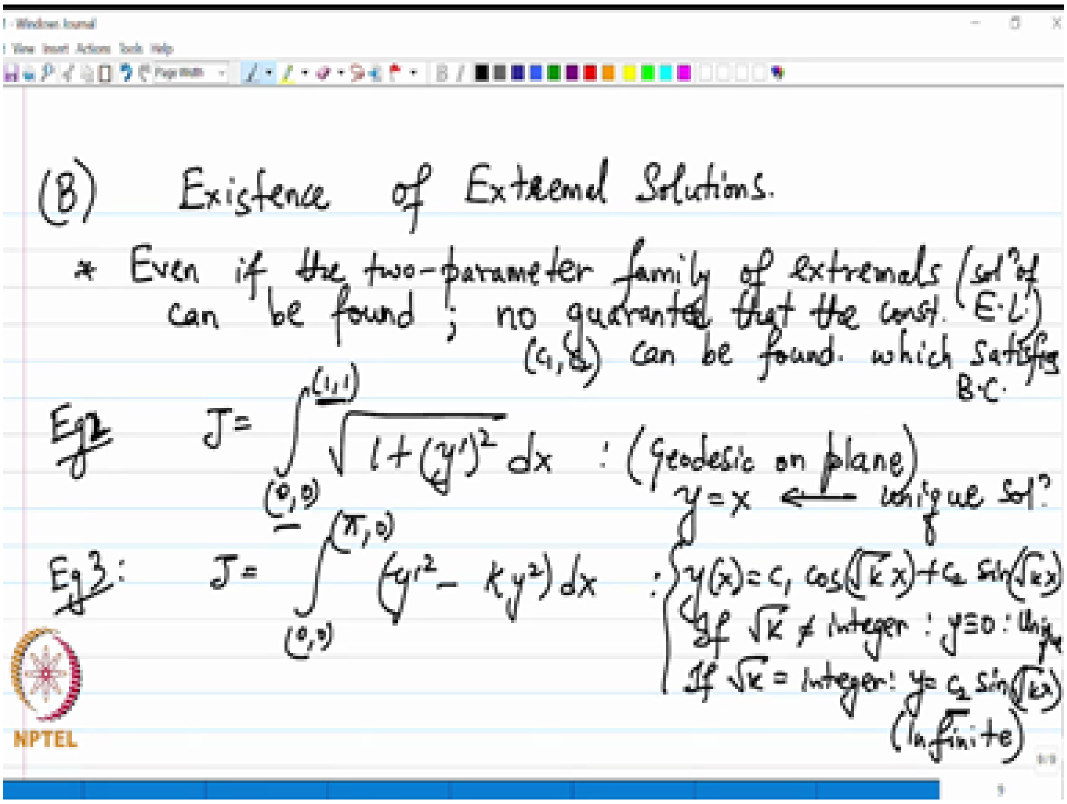

So, let us move ahead. So, then the second topic I want to discuss is about the existence of extremal solutions. So, well, just a brief motivation. So far we have just written Euler-Lagrange without even bothering about whether the solution exists or not, we have went ahead and solved the Euler-Lagrange equations in all the examples that we have seen so far.

However, we have not worried, whether even after we are able to solve the Euler-Lagrange, whether the solution makes any sense or not or whether the constants in in the solution, in the family of solutions that we get in Euler-Lagrange there they do exist any constraints or not which satisfies the boundary condition. So, what I just said is the following: Even if the two-parameter, even if the two-parameter family of extremals are, which are the solutions of my Euler-Lagrange equations can be found, we are able to integrate, well, there is no guarantee, there is no guarantee that the constants C_1 and C_2 can be found; there is no guarantee that the constantC_1 and C_2 can be found which satisfies the boundary condition, which satisfies the boundary condition.

So, let us look at an example to highlight what I just said. Let us go back to our example for the geodesic on a plane. Well, this is my second example of this lecture series.

Example 2: J $=\int_{(0,0)}^{(1,1)} \sqrt{1+{y'}^2} dx$, this is the geodesic problem or the problem of the shortest path and this is the geodesic on plane problem.

And we get that the solution is y = x for this problem, which was a unique solution satisfying the boundary condition given by these points on the integral. However, let us look at another example, which was also done in our previous lecture. Notice this particular case

$$J = \int_{(0,0)}^{(1,1)} ({y'}^2 - ky^2)dx$$

and we know that the solution to this problem was

$$y(x) = C_1 \cos\sqrt{K}x + C_2 \sin\sqrt{K}x$$

and we have further seen that the solution is going to exist such that it satisfies the boundary condition and there would be two cases. If your $\sqrt{k} \neq$ integer then y identically 0 is the unique solution that we have.

And we also saw that if $\sqrt{k}$ = Integer then then y = $C_2 \sin\sqrt{K}x$, giving us infinitely many solutions depending on this family of parameters C_2,so we could either have a unique solution or infinitely many solutions.

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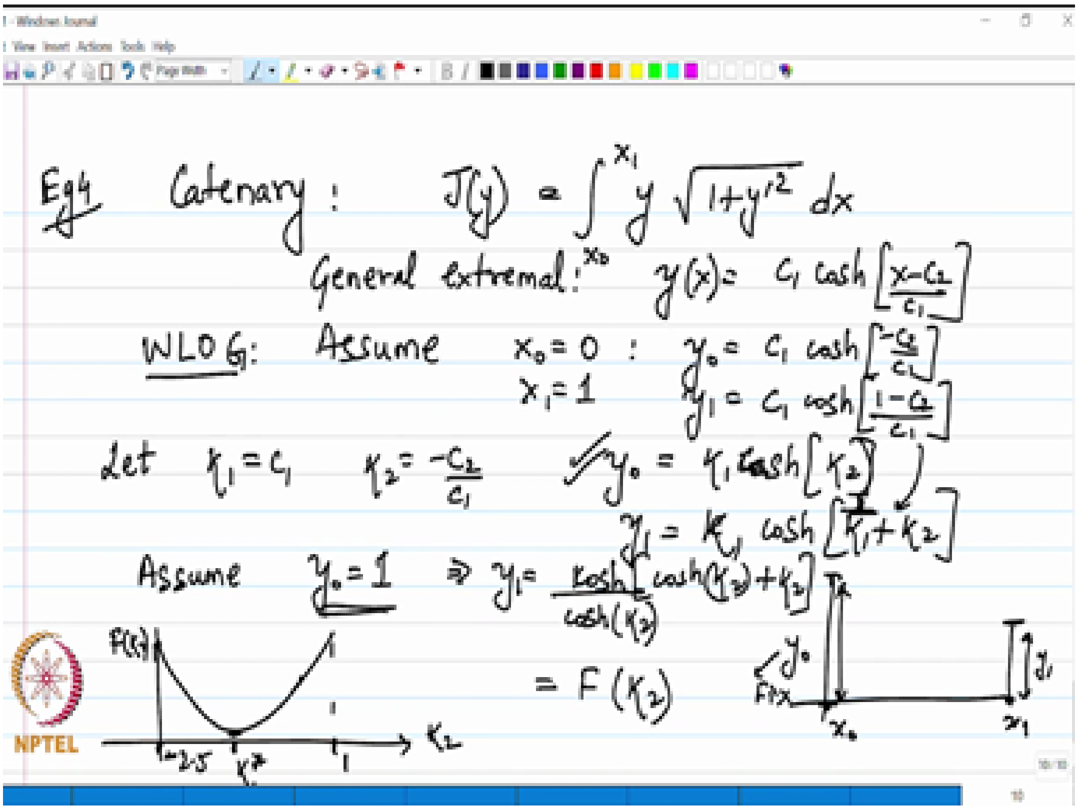

Now, let us look at another example that we have seen in the past lecture. An example of the catenary $J(y) = \int_{x_o}^{x_1} y\sqrt{1+{y'}^2}dx$, we know that the general extremal of this catenary problem is $y(x) = C_1 \cosh\left(\frac{x-C_2}{C_1}\right)$ We have just found out the solution to this problem in two lectures back and let us now further look at a class of this solution.

So, without loss of generality, let us assume that our boundary condition $x_o = 0$ and $x_1 = 1$. So, in that case my $y_o = C_1 \cosh\left(\frac{-C_2}{C_1}\right)$ by putting $x_o = 0$ and $y_1 = C_1 \cosh\left(\frac{1-C_2}{C_1}\right)$ by putting $x_1 = 1$ So, then the next stage to look at a simplified version of the solution is to change the set of constraints.

let us now introduce another set of two constraints, $\kappa_1 = C_1$ and $\kappa_2 = -\frac{C_2}{C_1}$. So, in that case my $y_0 = \kappa_1 \cosh\kappa_2$ and $y_1 = \kappa_1 \cosh\kappa_1 + \kappa_2$, again if we go back to the catenary problems, here we have the coordinate x_o and x_1 we have fixed that, Let us also further fix y_o and we are going to describe y_1 in terms of y_o .

let us assume $y_o = 1$ So, y_1 will be described in terms of y_o, if we do that, notice that

$$y_1 = \frac{\cosh[\cosh(\kappa_1) + \cosh(\kappa_2)]}{\cosh(\kappa_2)} = F(\kappa_2)$$

So, $\frac{1}{\kappa_1} = \cosh(\kappa_2)$ that comes from this particular expression here by plugging in $y_o = 1$. So now we have expressed my solution y_1 purely as a function of κ_2, one constant function. Now, if we were to plot this function i, let us say we were to plot $F(\kappa_2)$ versus κ_2

And I am just showing the solution and approximate solution from κ_2 from a range minus 2.5 to 1, this plot has been found using standard softwares. It turns out that this curve has exactly one minima, let us call this as kappa star. So between minus 2.5 to 1, there is just one minima κ^* . So the question says, this question that we have to ask is, is there always a solution to this catenary problem? The answer is not necessarily, look at the case.

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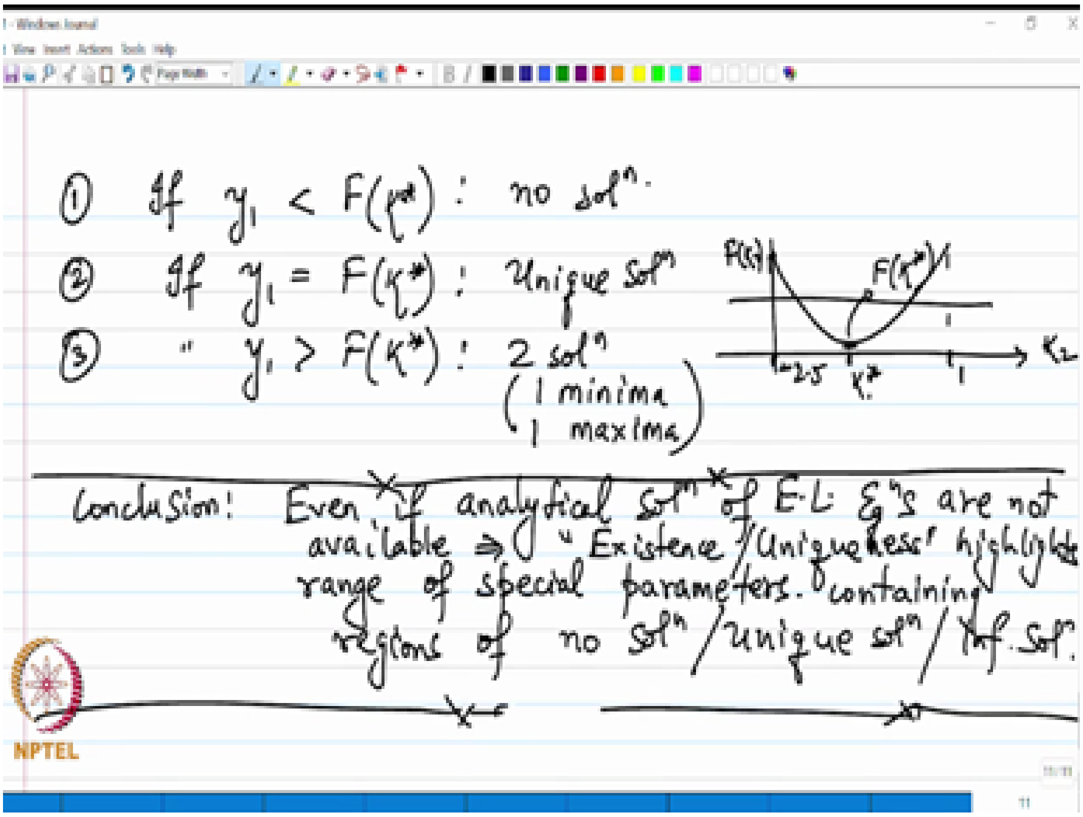

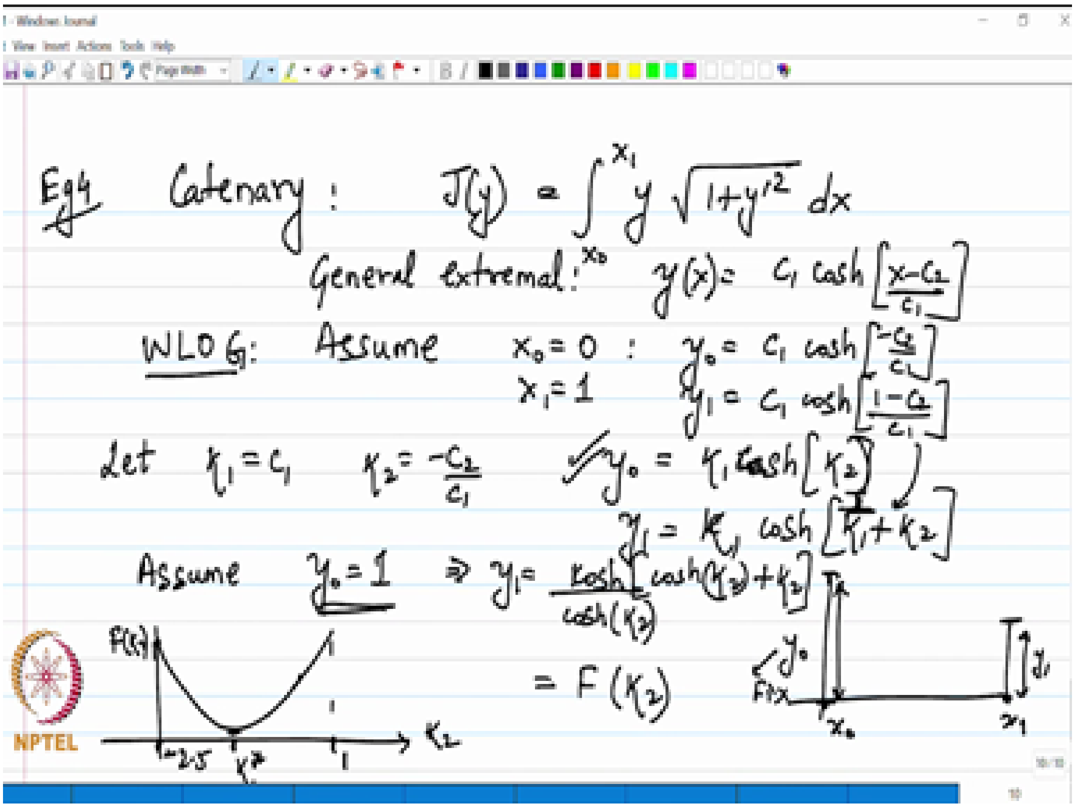

So, suppose look at case 1. So, suppose you have $y_1 < F(\kappa^*)$ So, notice that in this diagram if the solution falls below this value, I see that there will be no solution, y_1 will not be an external to the catenary problem.

On the other hand, if I have that $y_1 = F(\kappa^*)$ then I have a unique solution to my extremal problem given by the value of κ^*, the constant. On the other hand, if I have $y_1 > F(\kappa^*)$ we see that there are two solutions possible, one will be a minima and one will be a maxima.

So, we do not know which one is minima or maxima, but later on when we introduce the second variation or the sufficient conditions for the functionals, we will revisit this catenary problem to show that one of the solution is indeed a minima and the other is a maxima. So, the conclusion is as follows:

So, we conclude the discussion on this topic, and the conclusion we draw is as follows that even if our analytical solutions of Euler-Lagrange equations, analytical solutions of Euler Lagrange equations are not available, it turns out that the existence uniqueness, well, the existence uniqueness, uniqueness criteria highlights, the existence uniqueness criteria highlights the range of special parameters, highlights the range of special parameters containing regions of either no solutions or unique solution or infinitely many solutions. That is what we saw in the above example of the catenary.

So, at least we can find regions of the parameter space, where we have to search for our solutions to the Euler-Lagrange. So, existence uniqueness is also an important issue to consider while looking at the solution to the Euler-Lagrange equation.