Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 10 – Existence and Uniqueness of solutions - Part 01

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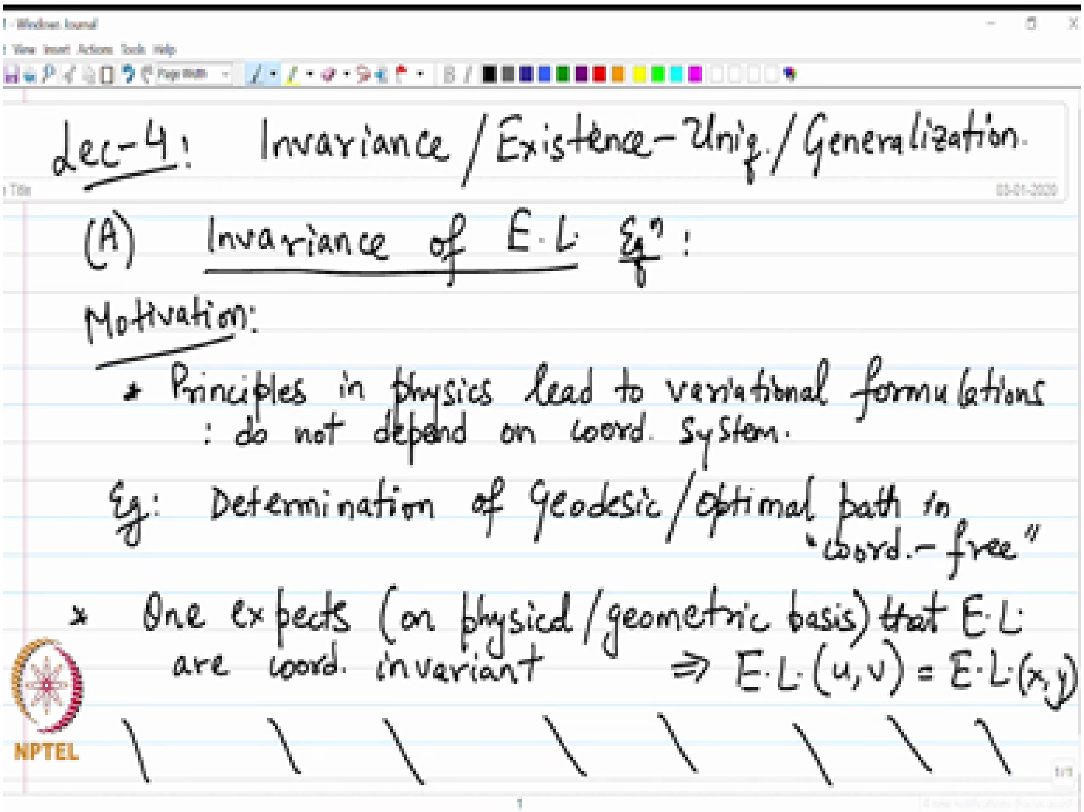

Good morning everyone. So in today's lecture, I am going to cover certain aspects of the solution of Euler-Lagrange equations, namely some of the aspects we have not covered so far yet. So, we have looked at the various cases or the various solutions of Euler-Lagrange equation for different case scenarios, but here are some of the issues that we have missed. Namely, the invariance issue or what happens to the solution of the Euler-Lagrange equation when we change the coordinate system of our solution.

So, how does Euler-Lagrange equation changes with a change in the coordinate system? So, that we will discuss in invariance and regarding whether the solution even exists or not is going to be discussed in the second topic and further we are going to generalize our Euler-Lagrange for more complicated problem. So, these are the three topics I am going to cover over this entire discourse of this lecture.

So, let us start with the first topic namely invariance, invariance of Euler-Lagrange equations, as I said invariance has everything to do with what happens to the Euler-Lagrange equation, how does it changes with the change in the coordinate system? Let us say from Cartesian coordinate to polar coordinate and so on, so forth. So, the motivation of this discussion is as follows:

We know all know that any physical phenomena or any physical principle should not and does not depend on the coordinate system of our choice. So, if we were to represent any physical system which is occurring in nature as a mathematical equation, that equation should be independent of the coordinate frame we are working, whether it is polar whether it is Cartesian it should not matter and which means that if we are solving an Euler-Lagrange equation for a physical phenomena, that Euler-Lagrange equation should also be independent of the coordinate system.

So, the motivation of this discussion as I just said is as follows: So, what we have is that principle in physics that lead to variational formulations, they do not depend on the coordinate system, we have numerous examples.

One example is the geodesics on a plane or the geodesic problem, the determination of geodesic is frame independent. Whether we want to find and later on we will show that when we find the geodesic on a sphere, it is sometimes better to find the geodesic using a polar coordinate rather than a Cartesian coordinate frame.

we will see that the geodesic comes out to be identical. So, the determination of the geodesic or the optimal path is coordinate free. This is one case which we will see in depth later on. So, based on this motivation one expects that the Euler-Lagrange equation should also be coordinate free because many of these Euler-Lagrange equations they describe the physical phenomena which are also coordinate free.

So, one expects on physical or geometric background, when expects on physical or geometric basis that Euler-Lagrange equations also are coordinate invariant. So, this is the expectation, this is the intuition that is provided to us and we will show that this is indeed the case in the form of a theorem later on. So, what I have just said is if I were to find the Euler-Lagrange equation, let us say with respect to any frame u, v, let us say variables u, v which represents another called coordinate frame, that should be the same Euler-Lagrange equation that we just described so far in the earlier lectures in the Cartesian coordinate frame.

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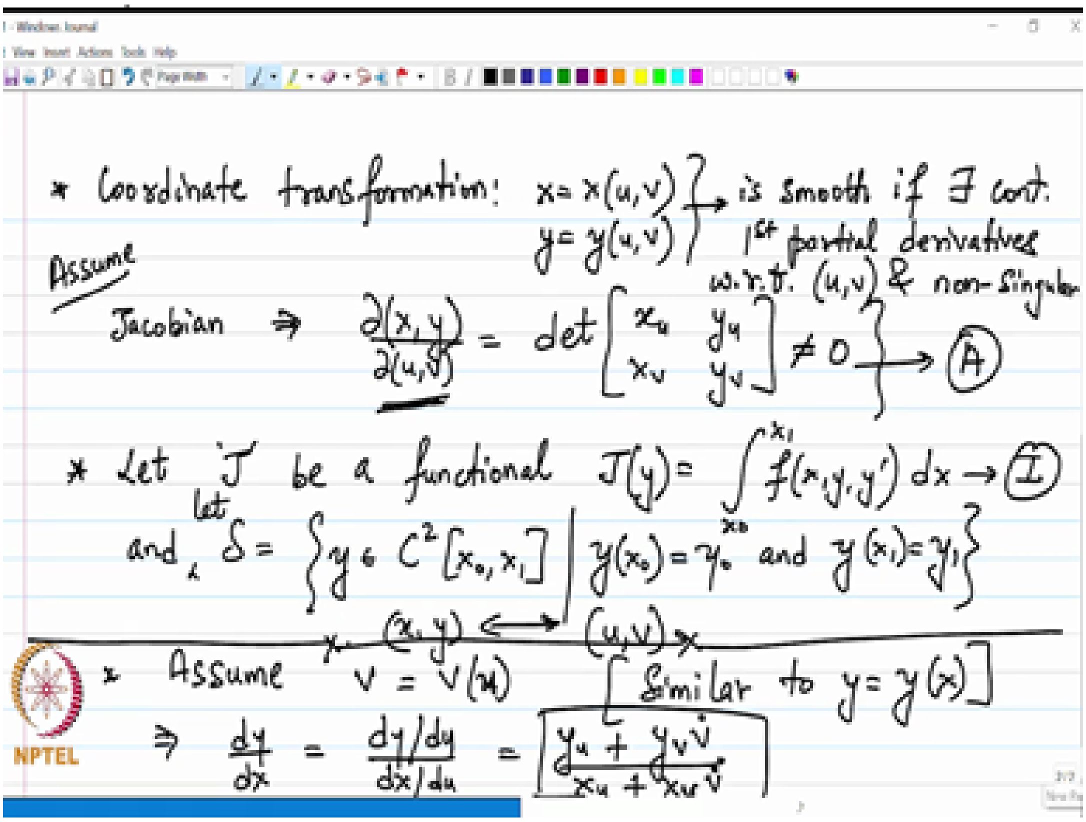

So, let us continue the discussion in a more rigorous fashion. So, let us look at a coordinate transformation where x = x(u, v) and y = y(u, v), coordinate transformation is smooth if there exists continuous first partial derivatives with respect to u, v and we have a non-singular Jacobian which is given by the evaluation of this following set

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \neq 0 \qquad \textbf{A}$$

So, the Jacobian should not be 0 only then we can guarantee that the transformation is smooth and orthogonal. So, that is what we assume. So, that is the underlying assumption.So, then let us look at a functional.
let J be a functional $\quad J(y) = \int_{x_o}^{x_1} f(x, y, y^{'})dx \qquad$ **I**

and Let $S = \left\{ y \in C^2[x_o, x_1] | y(x_o) = y_o \;\; and \;\; y(x_1) = y_1 \right\}$

We assume that this is the setup to describe the Euler-Lagrange equation in another coordinate frame u, v. So, this is the basic assumption and this is the setup that we have. Now, further let us look at the other set of assumptions that we need, we further assume so, we have two coordinate frames. We have x, y coordinate frame and we have the u, v coordinate frame.

So, we are trying to deduce the Euler-Lagrange equation in this frame in the u, v coordinate frame given the Euler-Lagrange in the Cartesian frame. So, further we assume that v is a function of u. Why we do that? Because if this assumption is similar to the assumption that y is a function of x, so, y is the dependent variable, which is dependent on the independent variable x.

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}} = \frac{y_u + y_u \dot{v}}{x_u + x_u \dot{v}}$$

All we have done is apply the chain rule and that gives us the derivative of y with respect to x in terms of u and v.

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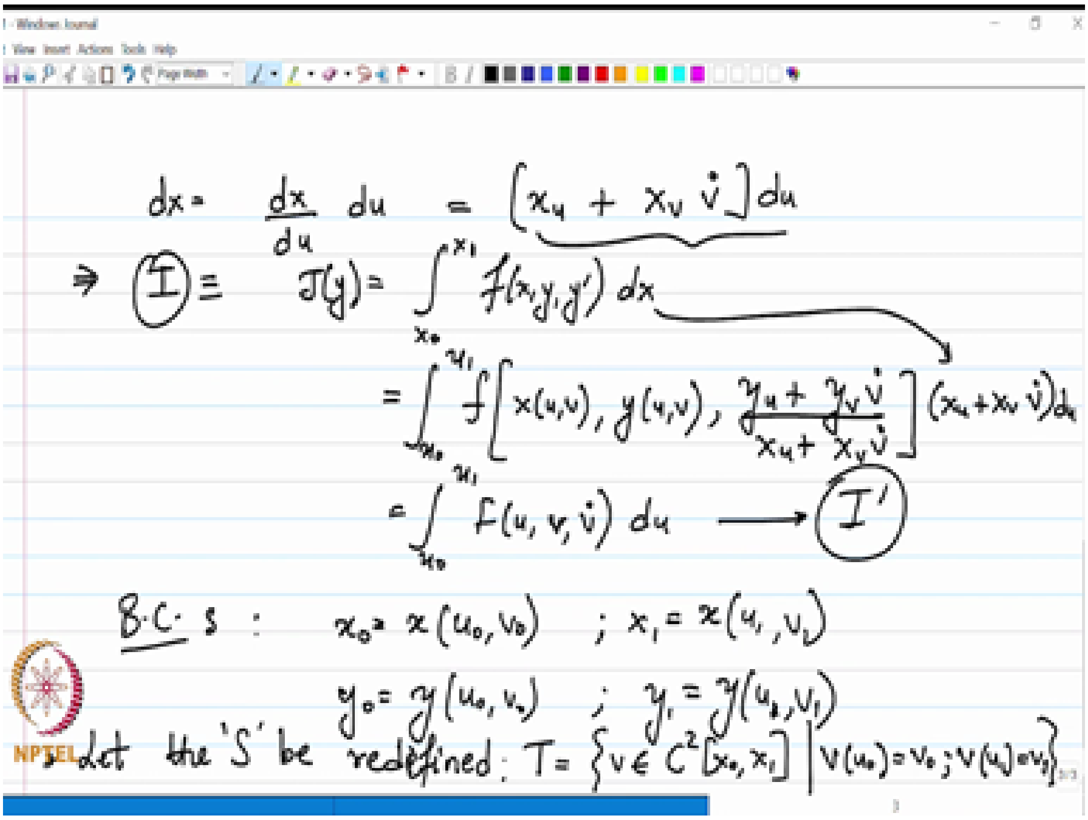

$$dx = \frac{dx}{du}du = [x_u + x_v\dot{v}]du$$

$$\Rightarrow I = J(y) = \int_{x_o}^{x_1} f(x, y, y^{'})dx = \int_{x_o}^{x_1} f\left[x(u, v), y(u, v), \frac{y_u + y_u\dot{v}}{x_u + x_u\dot{v}}\right](x_u + x_v\dot{v})du$$

$$= \int_{u_o}^{u_1} f(u, v, \dot{v})du \qquad\qquad \mathbf{I}^{'}$$

So, we have now finally reduced our functional completely in the form of the new coordinate system u and v. So, this is the analog of the functional in the Cartesian frame, then my new boundary conditions $x_o = x(u_o, v_o)$; $x_1 = x(u_1, v_1)$ and $y_o = y(u_o, v_o)$; $y_1 = y(u_1, v_1)$

So, let us further redefine our set S where the externals are going to come from $T = \left\{v \in C^2[x_o, x_1] | v(u_o) = v_o \ \ and \ \ v(u_1) = v_1\right\}$ now new set T is now the set of newly defined functions which are second order differentiable with this new set of boundary conditions. So, we are ready to describe the Euler-Lagrange equation, the equation in the new frame. So, what we have is the following; we ask the following question.

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Q. If $v \in T$ is an extremal of K (in (I')); then is $y \in S$ an extremal of J (vice-versa)?

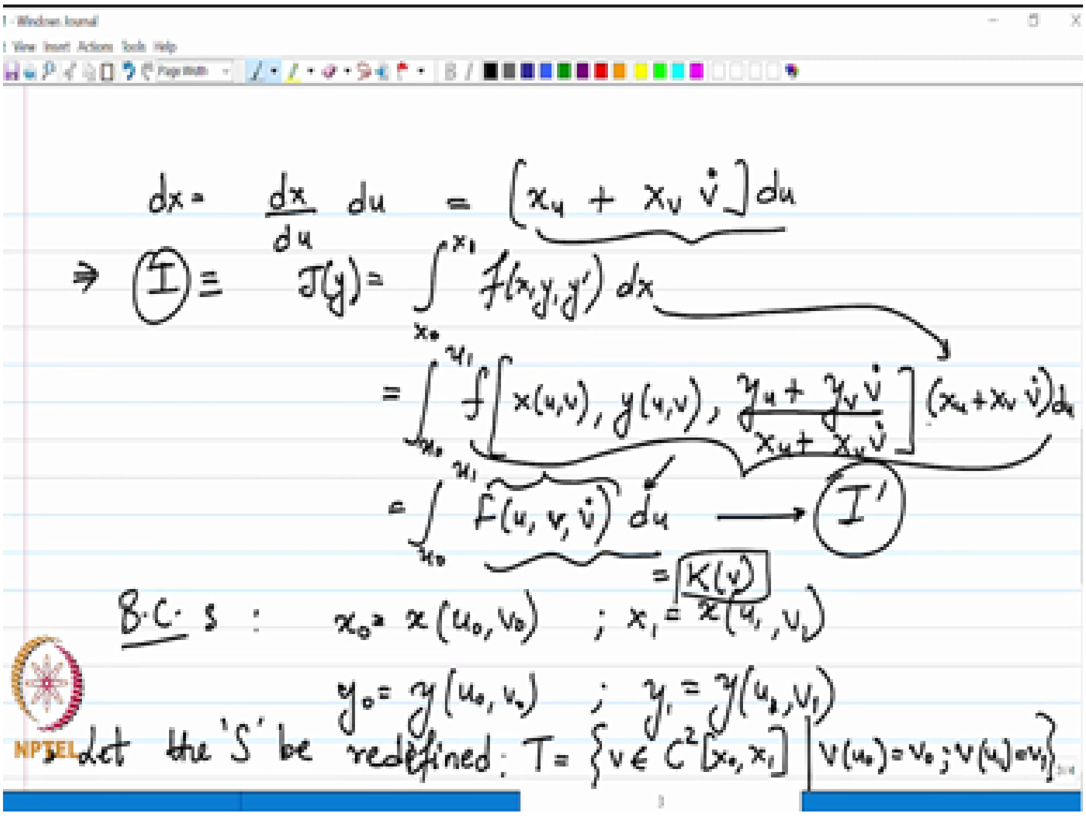
Thm 4: Let $y \in S$ and $v \in T$ be two fns. that satisfy the smooth non-sing. transformation (A). Then 'y' is an extremal for 'J' iff 'v' is an extremal for K.

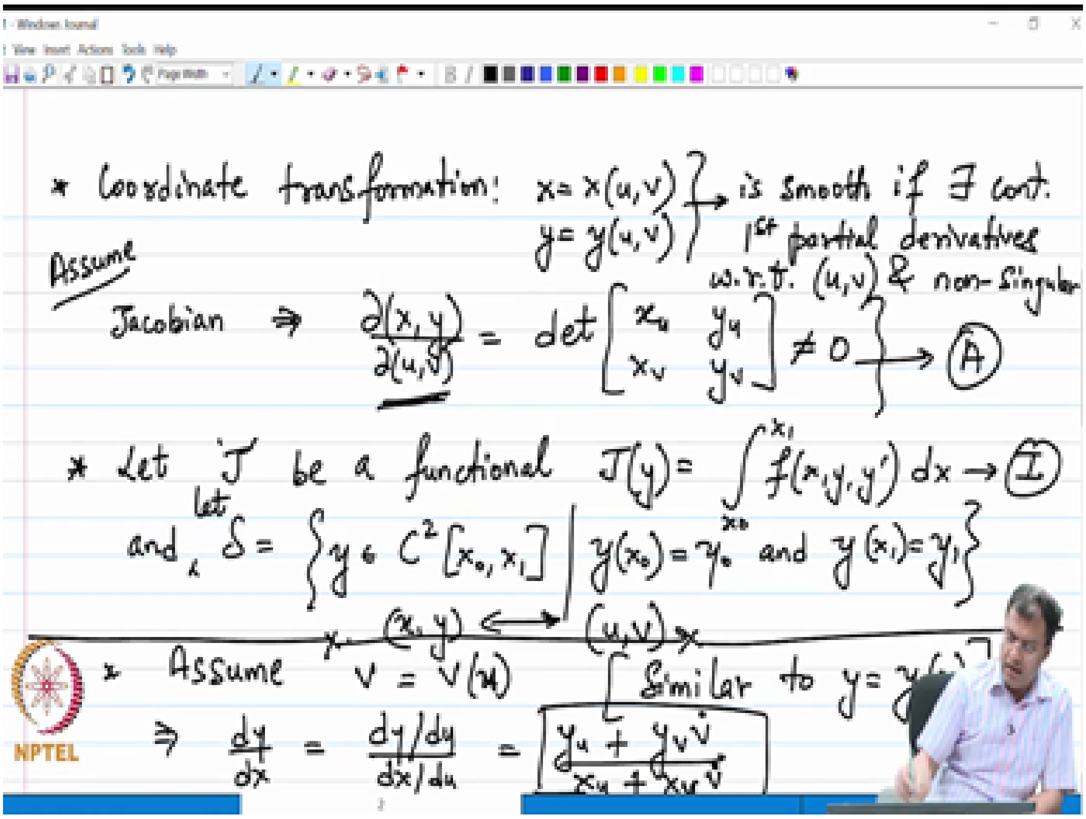
Proof: Suppose $v \in T$ is an extremal for K.

Then 'v' satisfies E.L. Eqn:

$$\frac{d}{du}\frac{\partial F}{\partial \dot{v}} - \frac{\partial F}{\partial v} = 0$$

$$F(u, v, \dot{v}) = f\left[x(u,v), y(u,v); \frac{y_u + y_v \dot{v}}{x_u + x_v \dot{v}}\right](x_u + x_v \dot{v})$$





The question that motivates our discussion is the following: So if v in this new set T is an extremal, Let me call this new extremal as K_v, my older extremal was defined as J (y). So, the question is, if v is an extremal of K described in $\mathbf{I}'$ then is y in S and extremal of the functional J and vice-versa. So, v is an extremal of K then is y an extremal of J and vice-versa.

So, we are going to answer this question in the form of a theorem that this the answer to this question is yes. So, that the result in the form of a theorem is as follows.
Theorem Suppose $y \in S$ and $v \in T$ be the two functions which satisfy the smooth, non-singular transformation A. A was a set of assumptions that we had, then y is an extremal for J if and only if v is an extremal for K. So, the answer to this question above that I posed is yes, via this theorem. And I am going to prove, the proof is very straightforward.

Suppose we assume that $v \in T$ is an extremall for the functional K then v must satisfy the Euler-Lagrange
$\frac{d}{du}\frac{\partial F}{\partial \dot{v}} - \frac{\partial F}{\partial v} = 0$ **1**
we see that this is the Euler-Lagrange being satisfied in this new coordinate frame.

So, that is my assumption. Now, I know that my f, let us go back a slide. Notice that my f here is nothing but this whole quantity times this whole quantity. So, f is the product of small f of this argument times this particular quantity in this bracket.

$$F(u, v\dot{v}) = f\left[x(u,v), y(u,v), \frac{y_u + y_u\dot{v}}{x_u + x_u\dot{v}}\right](x_u + x_v\dot{v})du$$

So, all I have to do now is to plug this f into this Euler-Lagrange and figure out all these partial derivatives in this boxed equation, right? So, let us do that step by step.

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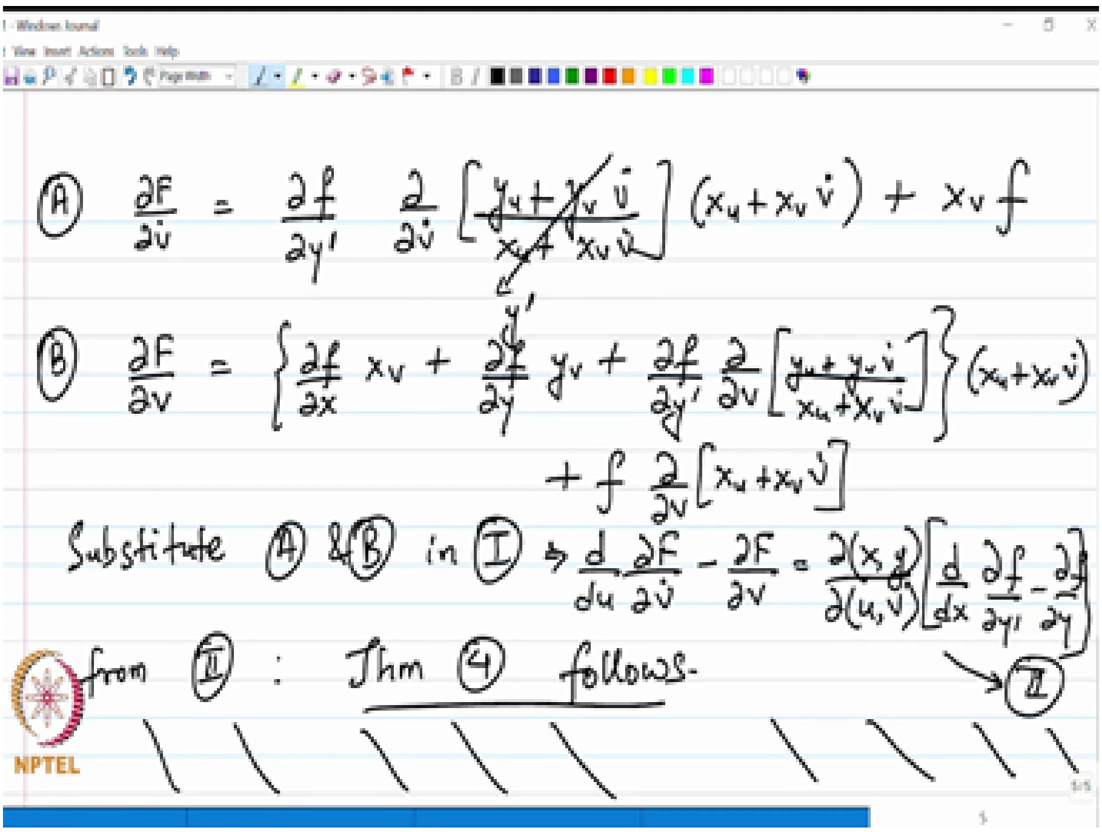

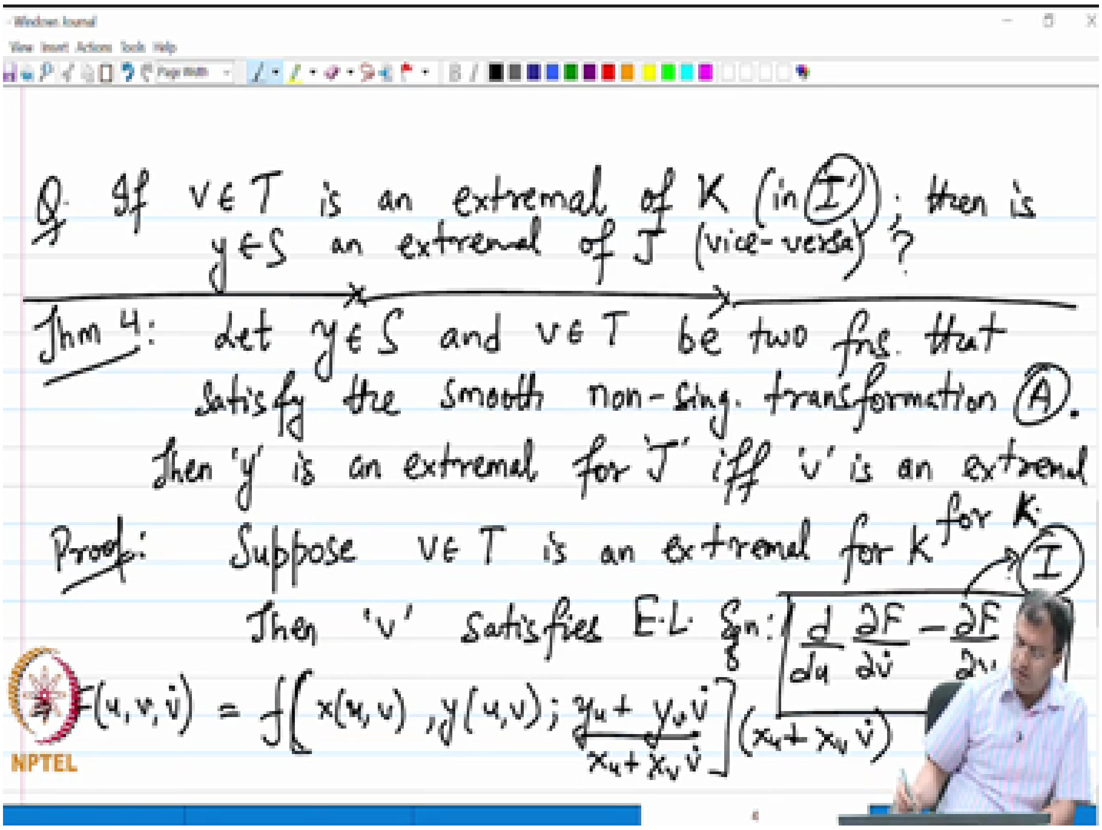

$$\textbf{A} \qquad \frac{\partial F}{\partial \dot{v}} = \frac{\partial f}{\partial y'}\frac{\partial}{\partial \dot{v}}\left[\frac{y_u + y_u\dot{v}}{x_u + x_u\dot{v}}\right](x_u + x_v\dot{v}) + x_v f$$

$$\textbf{B} \qquad \frac{\partial F}{\partial v} = \left\{\frac{\partial f}{\partial x}x_v + \frac{\partial f}{\partial y}y_v + \frac{\partial f}{\partial y'}\frac{\partial}{\partial \dot{v}}\left[\frac{y_u + y_u\dot{v}}{x_u + x_u\dot{v}}\right]\right\}(x_u + x_v\dot{v}) + f\frac{\partial}{\partial v}[x_u + x_v\dot{v}]$$

Substitute **A** and **B** in **1** and then after all the simplifications, you see that our equation reduces to the following form.

$$\Rightarrow \frac{d}{du}\frac{\partial F}{\partial \dot{v}} - \frac{\partial F}{\partial v} = \frac{\partial(x,y)}{\partial(u,v)}\left[\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right] \qquad \textbf{II}$$

From **II**, we will just say that theorem 4 follows. As simple as that, because if this quantity on the left hand side is 0 implies that the only this quantity on the bracket, the bracketed quantity on the right hand side will be 0 because the Jacobian term can never be 0; otherwise the coordinate transformation will be singular.

So, the Jacobian is non-zero implies theorem 4 follows, which means that Euler-Lagrange equation is coordinate independent or coordinate free.