Real Analysis - I

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Lecture – 27.1
Sets of Measure Zero

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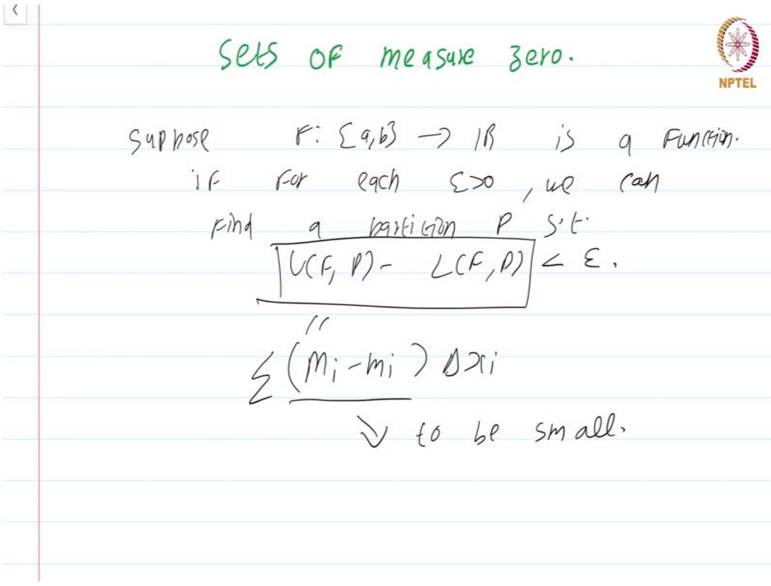

Suppose $f: [a, b] \to R$ is a function; we want to understand if f is integrable or not. We already know that f will be integrable, if for each $\varepsilon > 0$; we can find a partition we can find a partition P, such that $U(f, P) - L(f, P) < \varepsilon$

Now, from the way in which these upper sums and lower sums are defined, it is clear that this quantity$U(f, P) - L(f, P) < \varepsilon$ if and only if on each sub interval determined by the partition the variation of the function f is very small.

In other words, if you just rewrite this as $(M_i - m_i)\Delta x_i$; we want this quantity to be small, we want this quantity to be small. If this quantity is always large, when it is does not seem like it would be possible to make this entire summation $(M_i - m_i)\Delta x_i$ to be small, ok.

So, the one reasonable intuition behind what an integrable function is that the variation in the function on sub intervals is not too large. Now, this is seen clearly when we showed that any continuous function will automatically be Riemann integrable. Continuous functions behave very well; if you make these partitions thus, each interval in a partition really small, when it is reasonable to believe that $M_i - m_i$ can be made very very small and this was captured by using uniform continuity in the proof.

Now, we want to see whether there are other functions, apart from continuous functions that are Riemann integrable. In other words, we want to see functions for which this variation $M_i - m_i$ is not too large.

So, what we are going to see eventually in the next few modules is that, the precise condition under which f would be integrable is if the set of discontinuities or in other words the set of points where the oscillation of the function f is greater than 0; if that set is small, then the function will be Riemann integrable.

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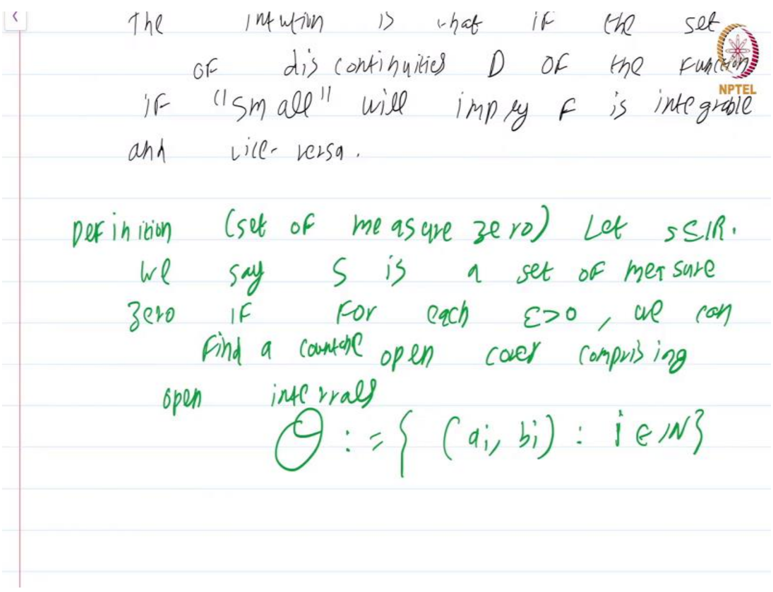

So, let me write that down the intuition is that if the set of discontinuities, that is the bad points, set of discontinuities D of the function f is small will imply f is integrable and vice versa. This is something that is palatable. Now, the only problem is this small that I have written should actually be in codes, because what exactly is a small set.

There is a precise technical condition called being a set of measure zero that captures this. Definition, this is the definition of set of measure zero; this will intuitively capture the fact that a set S is a small set.

Let $S \subseteq R$, we say, we say S is a set of measure zero; if for each $\varepsilon > 0$, we can find an open cover, rather a countable open cover, comprising open intervals.

Let us call this some fancy $\mathcal{O}$; this is just a collection of open intervals (a_i, b_i), such that i comes from the natural numbers. So, for each $\varepsilon > 0$; we can find a countable open cover comprising open intervals, such that the net length $\sum b_i - a_i < \varepsilon$, ok.

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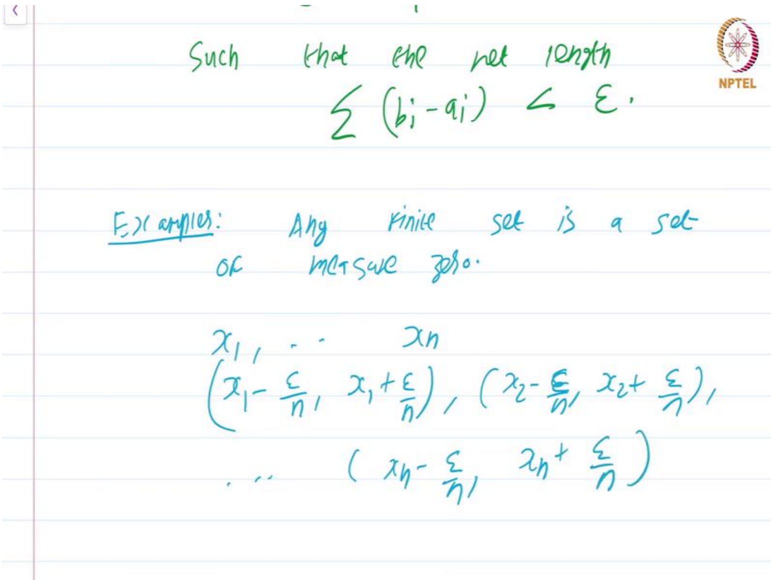

So, a set is a set of measure zero, if you can do this for each $\varepsilon > 0$; that means in some sense, the net length of this interval is smaller than any given ε .

Therefore, the net length of this interval which we use the more technical term measure, the net measure of this not this interval of this set S, the net measure of this set S can be made less than any given ε ; therefore it is sort of like having no measure at all, it is a set of measure zero, it is a small set.

So, this is a technical concept. So, it is good to see examples and many of them. Examples, any finite set is a set of measure zero. What I will do is, I will keep making these examples more and more complicated; it will turn out that I think the final example will subsume everything else, but it is a good idea to progress this way, any finite set is a set of measure zero.

How do you see this? Well, you enumerate the points as $(x_1, .., x_n)$ and just consider $\left(x_1 - \frac{\varepsilon}{n}, x_1 + \frac{\varepsilon}{n}\right)$. This has one of the elements of the cover and $\left(x_2 - \frac{\varepsilon}{n}, x_2 + \frac{\varepsilon}{n}\right) \dots \left(x_n - \frac{\varepsilon}{n}, x_n + \frac{\varepsilon}{n}\right)$, ok

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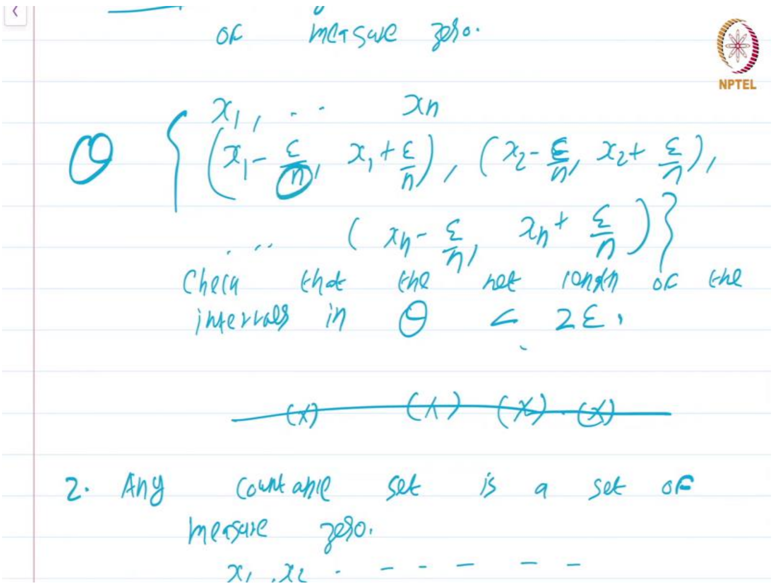

Now, the way I have done it, you will get an additional factor of 2; but I want you to check I want you to check that, the net length that the net length. So, let me put this as the set $\mathcal{O}$, the net length of the intervals in $\mathcal{O}$ is less than 2ε , I believe ok, please check this. So, that really does not matter; the factor of 2 really does not matter.

So, any finite set I can just; I can just look at this finite set and keep making these covers; I mean these open intervals around this finite set smaller and smaller to make sure that the net sum is less than ε this can always be done.

So, any finite set is automatically going to be a set of measure zero. Enhancing this any countable set any countable set; so let me call this 2, let me call this 1, any countable set is a set of measure zero.

This is also not too hard to see; what you do is, you enumerate this as $x_1, x_2 \ldots$, you make it the list, because it is a countable set, you can list it. Now, the problem is, there is no factor of n in the denominator that you could do to normalize it.

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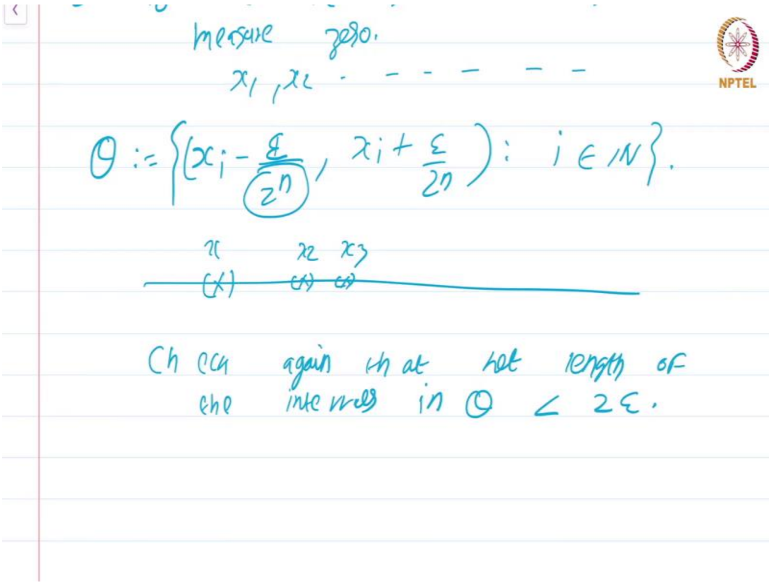

Well, that really does not matter; we are now experts in convergent series. What you do is, you consider the collection $\mathcal{O} = \{\left(x_i - \frac{\varepsilon}{2^n}, x_i + \frac{\varepsilon}{2^n}\right) : i \in N\}$. You look at intervals of this type. So, what is happening is, since we have no control over the denominator, we cannot have an $\frac{\varepsilon}{n}$; what you do is, you start shrinking subsequent things.

So, if this is x_1, this is x_2, this is x_3; you make the interval smaller and smaller and you control the size of these intervals by putting this 2^n in the denominator. We already know what $\sum \frac{1}{2^n}$ is and check again that net length of the intervals in $\mathcal{O} < 2\varepsilon$, I believe, ok.

Please check this in detail. So, this trick is very useful, it will be used many times; whenever you have some countable object and you want to control it, you put a $\frac{\varepsilon}{2^n}$, factor somewhere and that will allow us to control the sizes, excellent.

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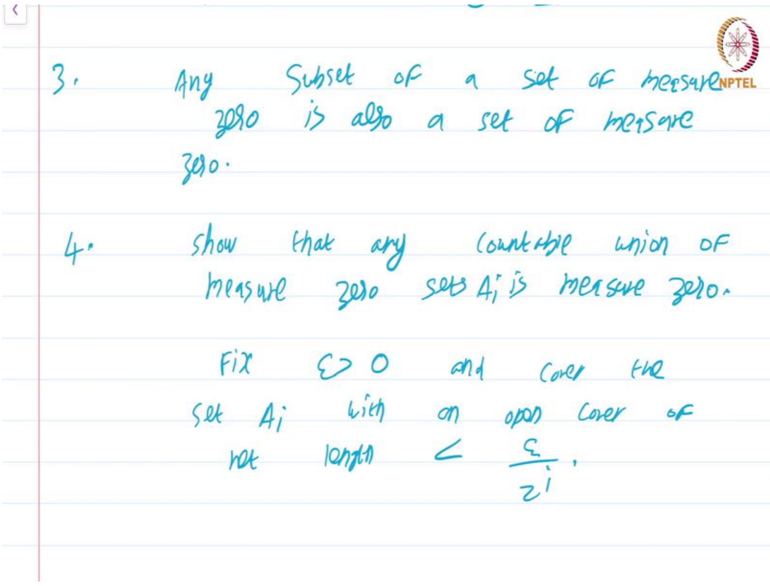

The 3rd example is now rather obvious; any subset of a set of measure zero is also a set of measure zero. This is fairly obvious, because what you do is, given an ε; you just consider the corresponding cover $\mathcal{O}$ for the larger set, the same thing will be a cover for the smaller set. So, this will work.

4th example, I want you to work out on your own. Show that any countable union of measure zero sets is measure zero.

If you take small sets and take countably many of them, you will still get back a small set. Let me give you a hint. So, you have to fix ε > 0 and produce a cover of the union, right. What you do is, you produce an open cover for each set in this union; for the first set, you make the net length less than $\frac{\varepsilon}{2}$, for the second set you make it less than $\frac{\varepsilon}{4}$, for the third set you make it less than $\frac{\varepsilon}{8}$.

You see where this is going, this is exactly like the countable set case; in fact this example subsumes the countable set case. So, fix $\varepsilon > 0$ and cover and cover the set A_i. So, any countable union of measure zero sets A_i is measure zero. So, cover the set A_i with an open cover; with an open cover of net length less than $\frac{\varepsilon}{2^i}$; this will give you the desired result, ok.

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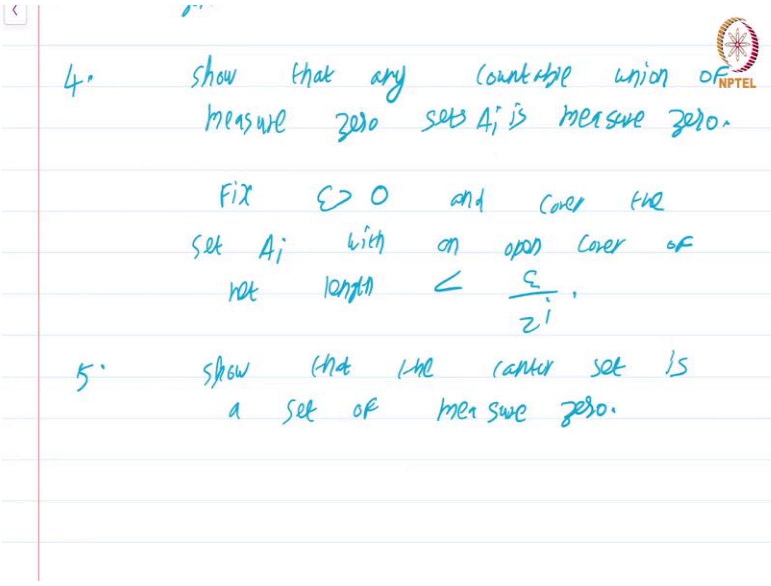

So, this is just elaborating the details; the key idea is already subsumed in the proof of countable set is going to be a set of measure zero. Another exercise for you; show that the cantor set the cantor set is a set of measure zero, ok. So, recall that the cantor set is uncountable. So, this sort of says that, do not be misled into thinking that only countable sets are going to be sets of measure zero; you have sets like the cantor set which are also sets of measure zero.

In this regard of the cantor set there is something interesting that is happening; we already saw that the net length of the intervals that were removed from $[0,1]$ is 1 right, we already saw that. So, it is sort of expected that the cantor set will be a set of measure zero.

So, the key thing is sets of measure zero, it is a capturing the idea that a set is small. We are going to now show in the next module that a function is integrable if and only if the set of discontinuities is actually a set of measure zero. This is a course on real analysis and you have just watched the module on sets of measure zero.