Real Analysis - I
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Lecture – 26.2
Criteria for Riemann Integrability

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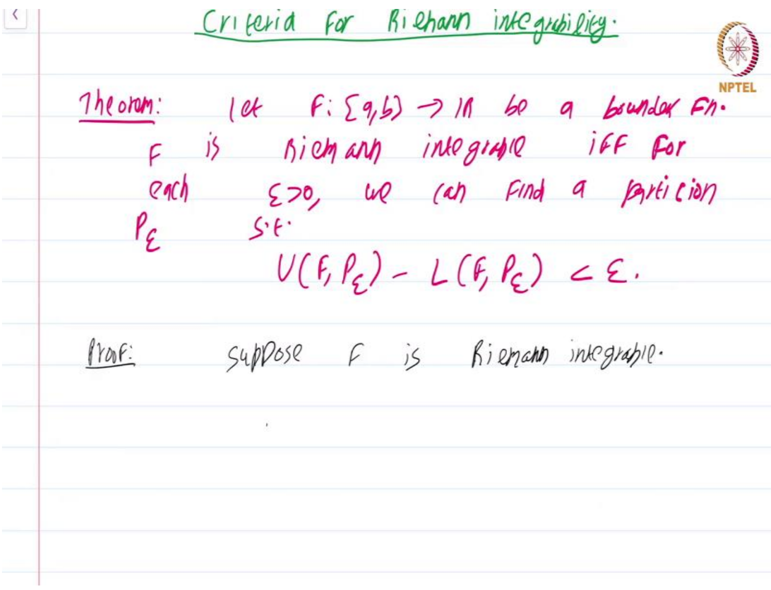

In this module we are going to study a simple Criteria for Riemann Integrability. A consequence of this criteria is the fact that any continuous function will be Riemann integrable. In a later module we shall study the Lebesgue integrability criteria that completely characterizes all Riemann integrable functions.

So, theorem, let $f: [a, b] \longrightarrow R$ be a bounded function, f is Riemann integrable, if and only if for each $\varepsilon > 0$, we can find a partition P_ε such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \varepsilon$. The proof of this is very straightforward. Proof: Suppose f is Riemann integrable, we will first deal with that direction.

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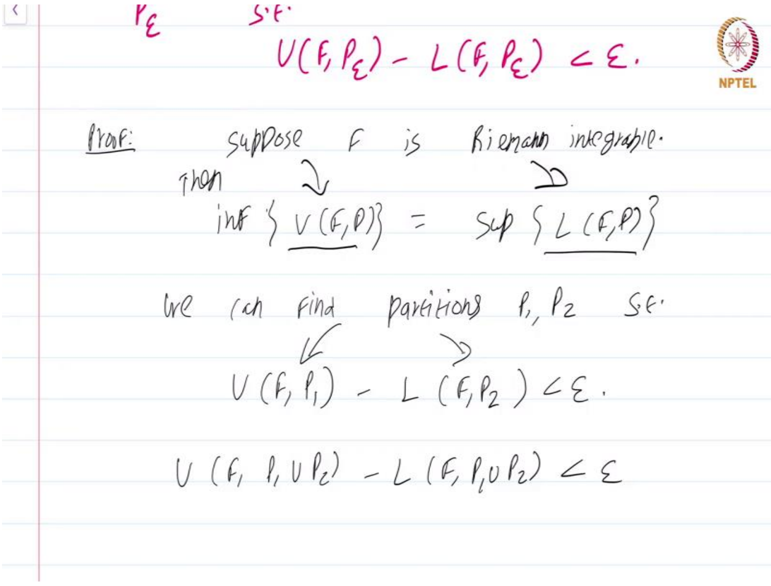

Suppose f is Riemann integrable, then that just means that $\inf U(f, P) = \sup L(f, P)$, that is the definition of a function being Riemann integrable. We just means that we can find partitions P_1, P_2, such that $U(f, P_1) – L(f, P_2) < \varepsilon$.

Why is this the case? Well this infimum is equal to this supremum that means, there must be some quantity here and some quantity here, that get arbitrarily closer and closer right. So, for this fixed ε we should be able to find a partition P_1 such and a partition P_2, such that $U(f, P\_1 - L(f, P\_2) <$ ε, ok.

Now, it is easy to find a common refinement. So, $U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) <$ ε why, because by going to the common refinement you are only going to decrease this quantity, sorry you are only going to increase this quantity and decrease this quantity. Therefore, the difference is only going to get lesser. So, you have $U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) <$ ε, ok.

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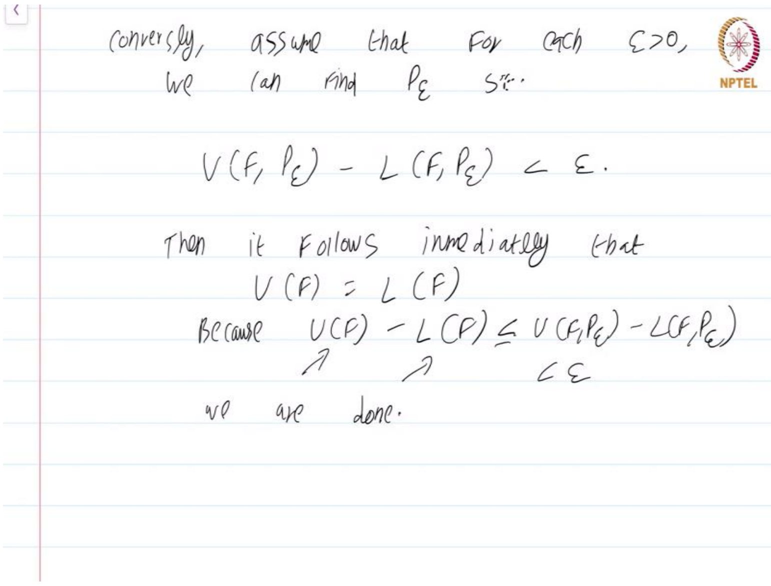

So, this proves one direction, conversely assume that for each $\varepsilon > 0$, we can find; we can find P_ε such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$. Then, it follows; it follows immediately, that $U(f) = L(f)$, why is this the case? Well, because $U(f) – L(f) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon)$, right.

Because, this is the infimum of the quantities $U(f, P_\varepsilon)$ and $L(f)$ is the supremum of the quantities. Therefore, $U(f) – L(f) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

So, what we have shown is $U(f)$ and $L(f)$ can be made ε close. Hence by a theorem, which we proved way back in week 1 or week 2, I think week 2 we are done, ok. So, because $U(f) – L(f) < \varepsilon$ we are done; we are done.

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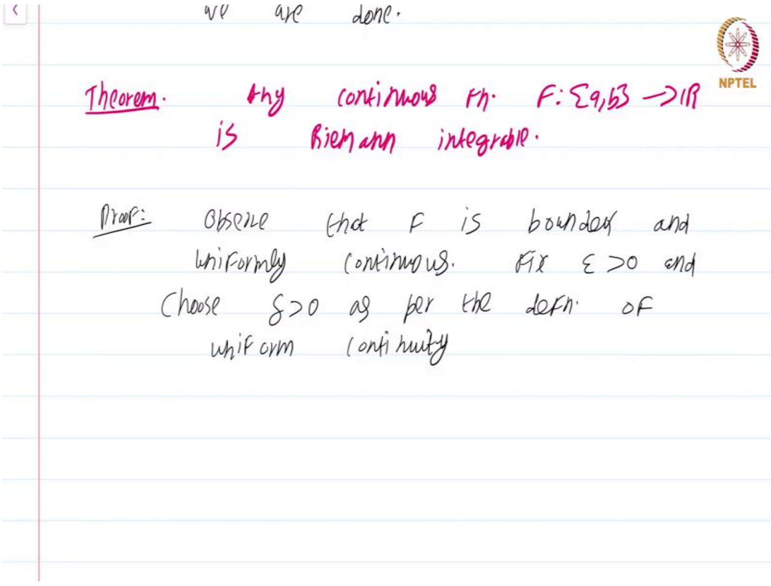

So, this was a rather simple criteria in fact, it just rephrases the fact that $U(f) = L(f)$, in a slightly different language. But, this is very useful as can be seen in the next theorem.

Theorem: Any continuous function $f: [ab] \longrightarrow R$ is Riemann integrable.

Now, here is a situation where we will not just use continuity, we will use uniform continuity in the proof. So, first of all observe that f is bounded because it attains its maxima and minima it is going to be bounded. We have already seen that the images has to be also going to be a closed interval. Observe that f is bounded and uniformly continuous, ok.

So, what we are going to do is, we are going to produce for each ε a partitionP_{ε}, such that the previous criterion is satisfied. So, fix $\varepsilon > 0$. And choose $\delta > 0$ as per the definition of uniform continuity. This just means, as a recall, this should be digested by you and flowing in your veins by now.

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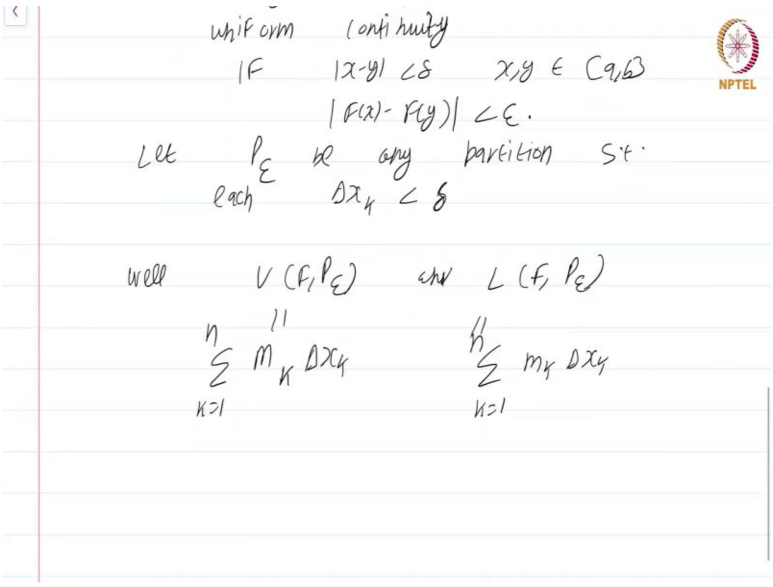

But, just to recall if $|x - y| < \delta$, $x, y \in [ab]$, then $|f(x) - f(y)| < \varepsilon$, ok. So, we have now got this condition of uniform continuity that sort of says that the moment points are close enough to each other then the values are going to be close to each other also.

Now, this will immediately show that we can find a partition that we want. Let P be any partition; be any partition or rather let P_ε be any partition such that each $\Delta x_k < \delta$. That just means that consecutive points in this partition are less than δ distance away, ok.

Now, what will this show? Well, let us compute $U(f, P_\varepsilon)$ and $L(f, P_\varepsilon)$, ok. This is just going to be $\sum M_{k_{\Delta x_k}}$ here, this x_k looks like a subscript Δx_k; k running from 1 to n. Recall capital M_k is the maximum and this is going to be $\sum_{k=1}^{n} m_k \, \Delta x_k$.

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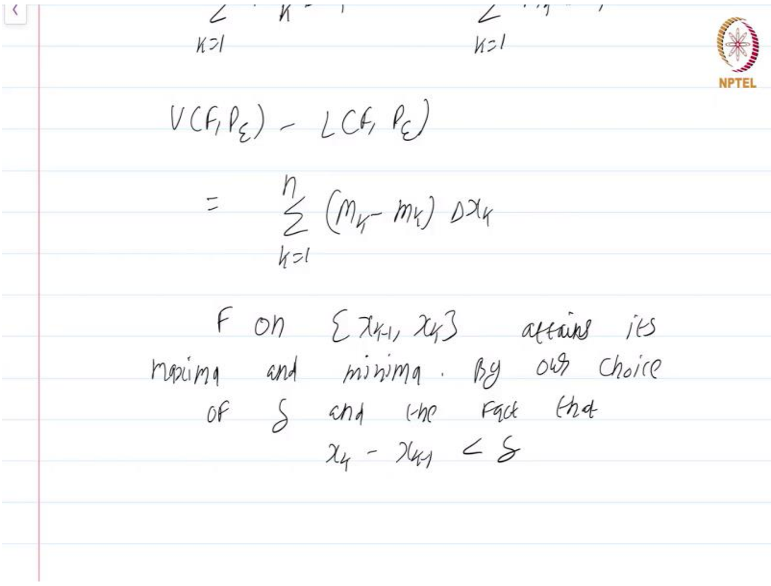

Now, we have this. What we are really interested in from the previous criterion is $U(f, P_\varepsilon) - L(f, P_\varepsilon)$, right. And a moments calculation will tell you that this is $\sum_{k=1}^{n}(M_k - m_k)\Delta x_k$, well, that was easy enough. Well, what do you know about $M_k - m_k$? Well, this is a continuous function; this is, f is a continuous function.

So, consider f on $[x_{k-1},, x_k]$ ok, this f attains its maxima and minima. That means, there are points in this interval $[x_{k-1},, x_k]$ and such that f of that point is capital M_k, and another point in this interval $[x_{k-1},, x_k]$ or rather I reverse the thing its $[x_{k-1},, x_k]$ sorry about that $[x_{k-1},, x_k]$.

There are points in this interval such that f of that point is capital M_k and f of that other point is small m_k. We can find points here simply because, this function is continuous and this is a compact set therefore, f will attain its maxima and minima, ok. But, that just forces, by our choice of δ and the fact that $x_k - x_{k-1}$ has to be less than δ, because, we chose the partition that way.

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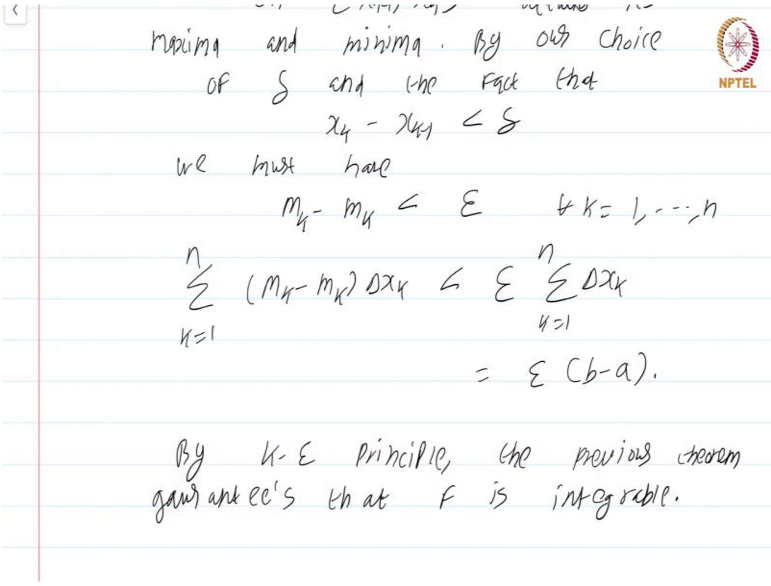

We must have; we must have capital $M_k – m_k < \varepsilon$. And the beauty of this fact is this is true for all k; all k from 1 to n, irrespective of the choice of k this is have to be this will have to be true. That just means $\sum_{k=1}^{n}(M_k - m_k)\Delta x_k < \varepsilon \sum_{k=1}^{n} \Delta x_k$.

And just thinking what this $\sum_{k=1}^{n} \Delta x_k$ is you will be able to see that this is just $\varepsilon(b – a)$. Now, what we have shown is given any ε we can find a partition P such that, $U(f, P_\varepsilon) – L(f, P_\varepsilon) < \varepsilon(b – a)$, by $K - \varepsilon$ principle; by $K - \varepsilon$ principle, the previous criterion previous theorem guarantees that f is integrable, ok.

So, this completes the proof; the proof was an application of several properties of continuous functions, please go through the proof carefully and see where you see; where each property of continuous functions were applied. To even begin the proof we had to use the fact, that continuous functions on compact sets will have to be bounded. We have defined integrals only for bounded functions.

This is a course on real analysis and you have just watched the module on criterion for Riemann integrability.