Real Analysis - I
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Lecture – 26.1
The Definition of the Riemann Integral

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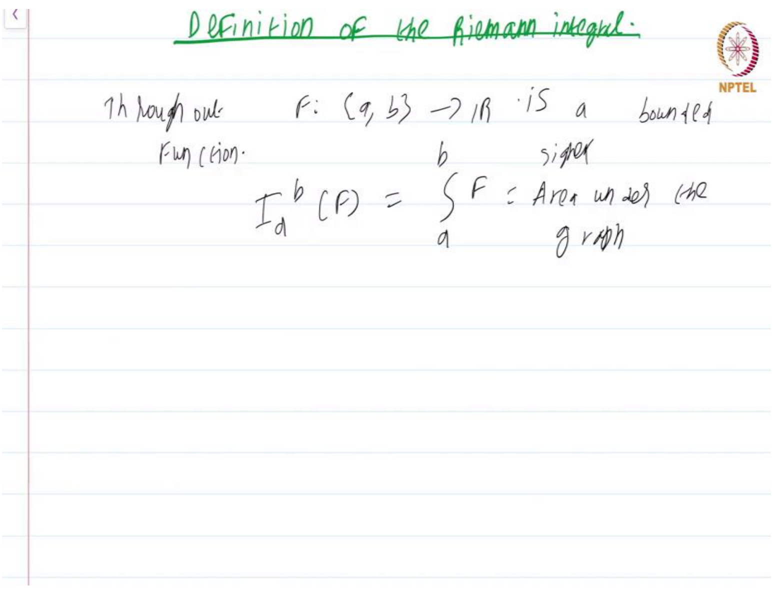

We shall define the Riemann Integral in this module. The approach we take is due to Darboux which is polished and refined version of Riemann's original construction. Without further ado, let us first set things up. So, throughout $f: [a, b] \longrightarrow R$ is a bounded function. Our objective is to assign a meaning to this $I_a^b(f)$ or in more common notation integral $\int_a^b f$.

This is supposed to be area under the graph, the signed area to be precise; the signed area under the graph. Now, we are going to take our approach motivated by the final axiom in the axioms of area that I listed; that is a figure can be assigned an area, if it can be exhausted both from the inside and from the outside by figures that are just adjacent rectangles. Now, we are going to take this approach; so I need to make some definitions.

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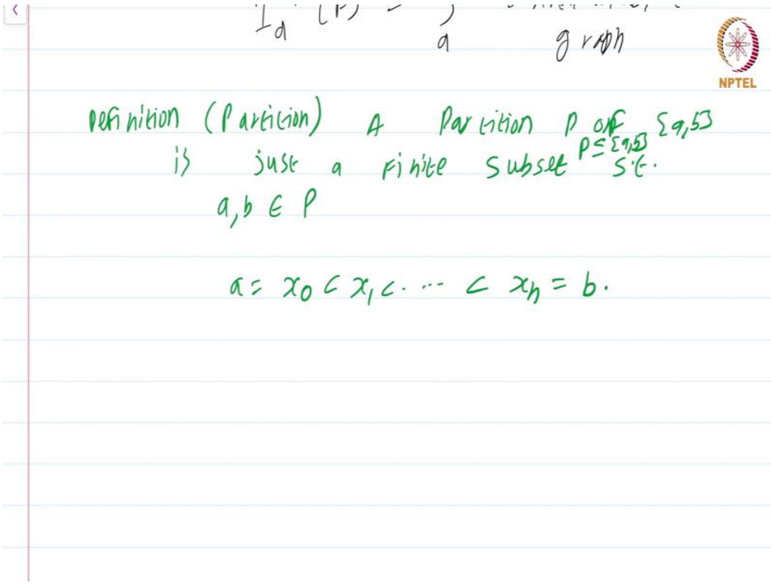

This is the definition of a partition. So, a partition P of $[a, b]$ is just a; finite subset such that $a, b \in P$. So, it is just a finite subset P of $[a, b]$. It is just a subset of $[a, b]$ that also happens to contain the points both a as well as b.

Typically, we list out the points of $[a, b]$ in this way. We list out a first and call it x_0 and it is usually listed this way and the final element x_n is going to be b ok. So, we usually list the elements of a partition in increasing order and this will make the notation of what is to come, really it is transparent and straightforward ok.

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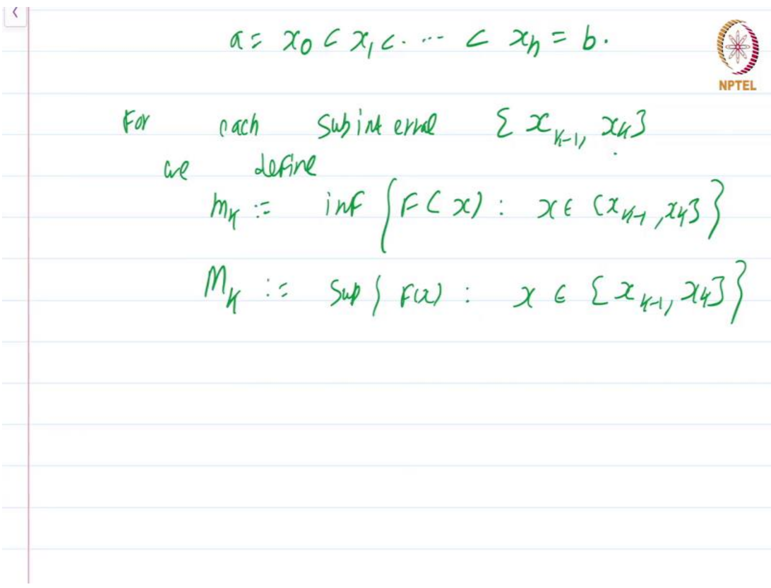

Now, for each sub interval; for each sub interval, $[x_{k-1}, x_k]$; we define the lower sum with respect to just a moment, first I need to tell you before, for each sub interval we define small $m_i$to be infimum of $f(x)$.

So, let me use precise notation; it is just $\inf\{f(x): x \in [x_{k-1}, x_k]\}$ and capital $M_k = \sup\{f(x): x \in [x_{k-1}, x_k]\}$ ok. So, these should actually be small m_k and capital M_k because k is the variable that I have used as the running index. So, on each sub interval you look at the supremum and the infimum of the values of the function f and call it small m_k and capital M_k.

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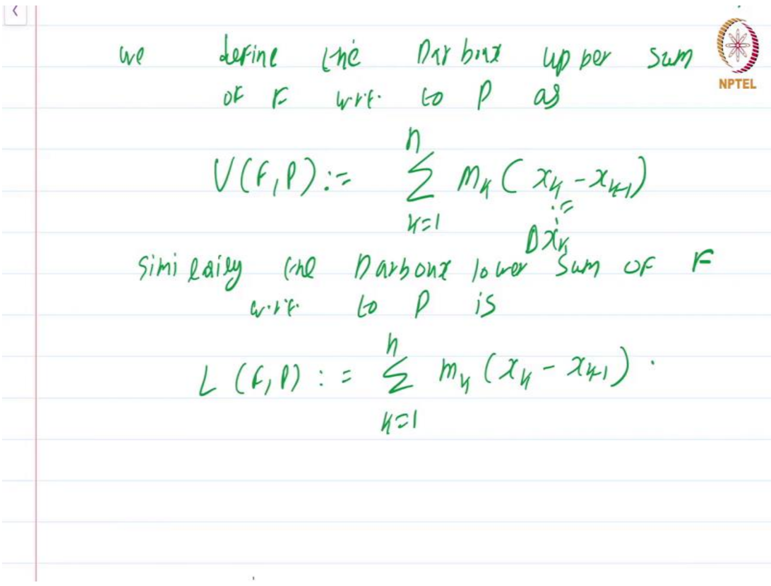

Now, we can define; we define the Darboux upper sum of f with respect to this partition P as $U(f, P)$ is by definition just the sum, as you run over all the sub intervals of M_k times the length of the interval, $U(f, P) = \sum_k M_k(x_k - x_{k-1})$, this $x_k - x_{k-1}$ quantity which is the length of the sub interval will occur so frequently that we will just abbreviate it as Δx_k ok.

Similarly, the Darboux lower sum of f with respect to P respect to P is $L(f, P)$is by definition just $L(f, p) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$, ok. So, these are the Darboux upper sums and the Darboux lower sums ok. What is what are we trying to capture with these sums?

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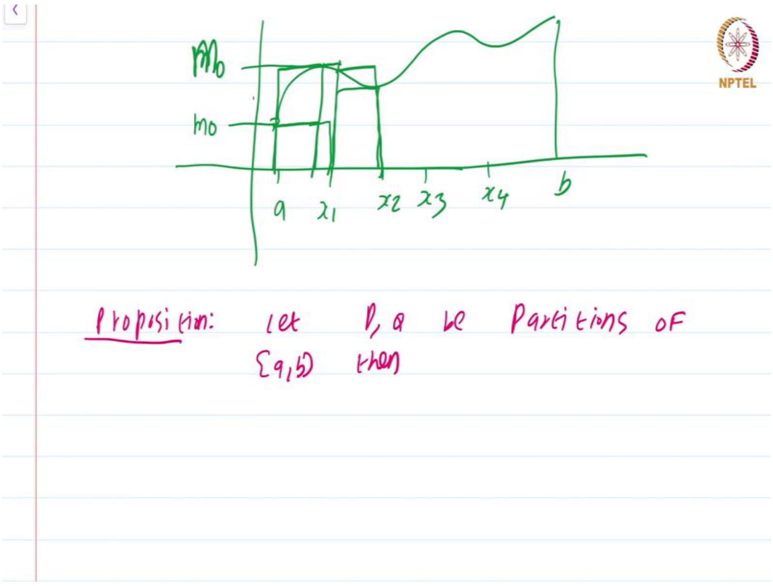

Well, let us just take a positive function so that things become easy when I draw areas. What we have done is this a, b; we have broken up into smaller intervals. So, here is x_1, x_2, x_3, x_4 and let us say the final one $x_5 = b$; of course, $x_0 = a$. Now, what we are doing is in each one of these intervals you are noting down which is the minimum point and which is the maximum point.

So, for instance this point will, be this value is m_0 and somewhere here; this is m_1, this is M_0, sorry, this is capital M_0, ok. So, what you do is you draw two rectangles. One with the one of the sides being of length small m_0, the other side is of course, of length I should go all the way. The other side is of length of course, $x_1 - x_0$which is just $x_1 - a$.

And then for the second; you use this maximum point, the length of the maximum thing that is capital M_0; so you draw the second rectangle ok. So, similarly you do for all the other sub intervals also. for instance, these sub intervals, this is sort of the minimum and the maximum is sort of this; I hope the picture is not too confusing.

So, essentially what this gives us is this figure that comprises adjacent rectangles, you will get two pairs; you will get a pair of figures; one comprising the maximum points and the other corresponding to the minimum points.

And it is intuitively clear that these adjacent rectangles formed by the taking the maximums will sort of contain the graph, the area under the graph of the function f; whereas, those corresponding to the minimums will sort of be contained within the graph; within the area under the graph, the region under the graph ok. So, this is sort of mimicking the idea in the final axiom of the definition of area ok. Now, we immediately have this very simple proposition; we have this very simple proposition.

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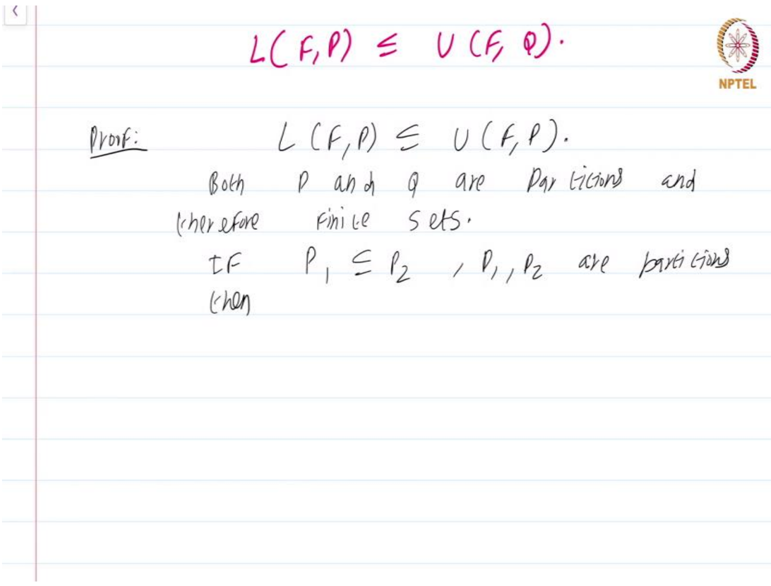

Let P , Q be partitions of $[a, b]$; partitions of the closed interval $[a, b]$, then this $L(f, P) \leq U(f, Q)$. No matter what partitions you take, the lower sum with any partition will always be less than or equal to the upper sum with any other partition; it does not really matter.

Proof; now, this should be obvious to you $L(f, P)$ is always less than or equal to $U(f, P)$; that simply follows straight from the definitions. At each sub interval, you are choosing the minimum values for $L(f, P)$; whereas, you are choosing the maximum values for $U(f, P)$.

So, this inequality that $L(f, P) \leq U(f, P)$ is obvious ok. Now, what we will do is both P and Q are partitions and therefore, finite sets, ok.

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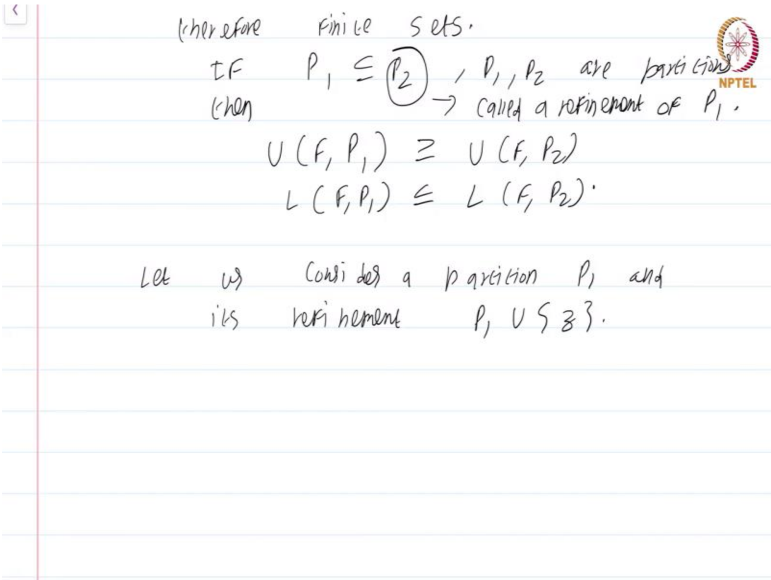

first, what we will show is if $P_1 \subseteq P_2$; P_1, P_2 are partitions; then $U(f, P_1) \geq U(f, P_2)$. And similarly $L(f, P_1) \leq L(f, P_2)$, ok. So, such a thing; this P_2 is also called a refinement; is called a refinement of P_1, for obvious reasons; the terminology here is very very intuitive and visual. If one partition contains another partition, we say that the larger partition is a refinement of the smaller partition.

Now, what this is saying is when you refine a partition; the upper sum has to decrease whereas, the lower sum has to increase. Why is this the case? Well, let us just see, as I remarked all these partitions are finite sets. Let us consider; let us consider a partition P_1 and its refinement $P_1 \cup \{z\}$. I am just going to add a single point z to a partition and let us see what happens in this case; this is the simplest scenario.

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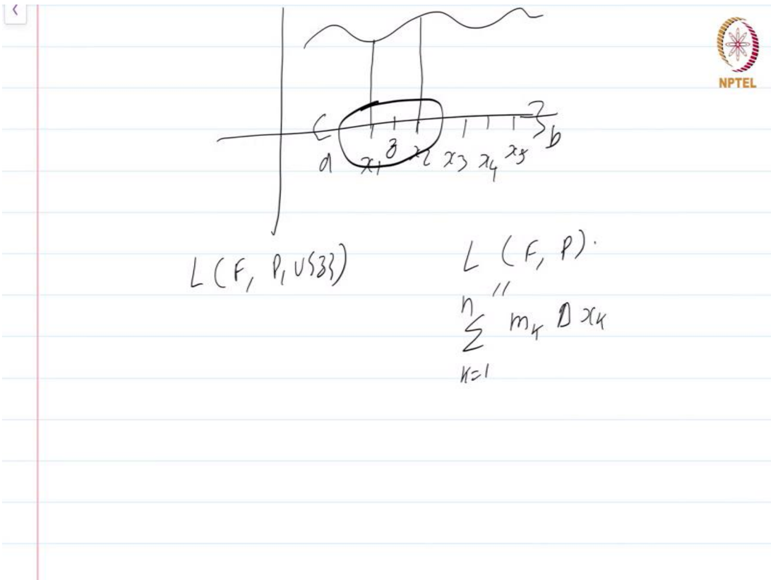

So, again let us draw the picture; we will draw a sort of sketch picture where not much of the situation is fully captured. So, you have $x_1, x_2, \ldots x_5$; let us say x_3, x_4. Let us say in the middle of this; you are adding this point z ok. So, we have these functions which is somewhere floating in the air.

What is happening is when you are computing; when you compute $L(f, P_1 \cup \{z\})$ ok, we have to compare this with $L(f, P)$ ok. Now, this $L(f, P)$ is just going to be $\sum_{k=1}^{n} m_k \, \Delta x_k$, this was our notation.

Now, what we will do is we will just focus on this particular sub interval, where all the action is happening; everywhere else nothing interesting is happening. Well, I can always rewrite; I can always rewrite. So, this is in our notation this is going to be Δx_2 right; in our notation, this is Δx_2 right.

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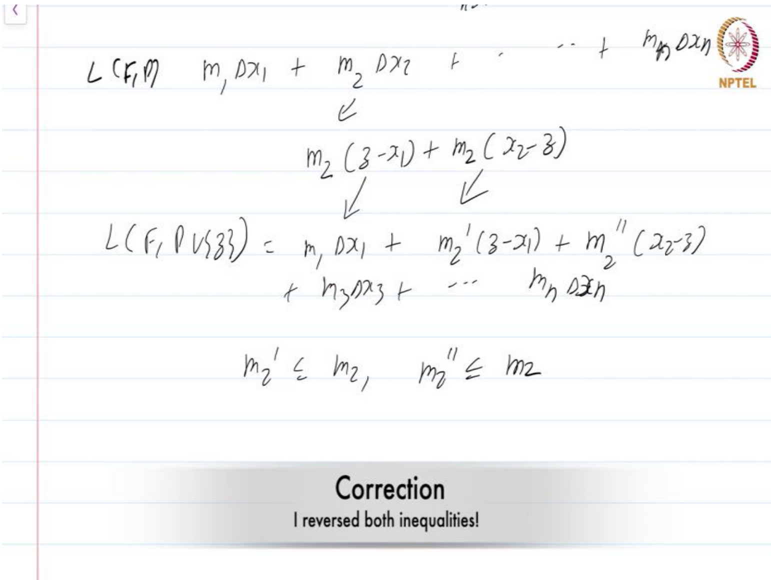

So, what is going to happen? Well, you have $m_1\Delta x_1 + m_2\Delta x_2 +$. I am not worried about the other terms and finally, you have $m_n\Delta x_n$. This $m_2\Delta x_2$. I can rewrite as $m_2(z - x_1) + m_2(x_2 - z)$ right. I am just rewriting $\Delta x_2 = (z - x_1) + (x_2 - z)$, that is all I have done.

Now, if you consider; so this is $L(f, P)$; now if you consider $L(f, P \cup \{z\})$, observe that all terms will be the same; all terms will be the same except these two terms; these two terms will possibly get lesser; why will it possibly get lesser? Well, it will possibly get lesser because you are taking the infimum on a smaller interval.

So, when you take infimum on a smaller interval; it was only going to increase ok. So, here you will have $m_1\Delta x_1 + m_2'(z - x_1) + m_2''(x_2 - z) + m_3\Delta x_3 + \cdots + m_n\Delta x_n$, right. And we have $m_2' \leq m_2$, $m_2'' \leq m_2$ ok.

So, this I have just sort of taken this point z to be in one of the intervals for concreteness; specifically, we have taken it in the second one $[x_1, x_2]$ interval, but it could have arose on anywhere; it could have been anywhere and the same argument would work.

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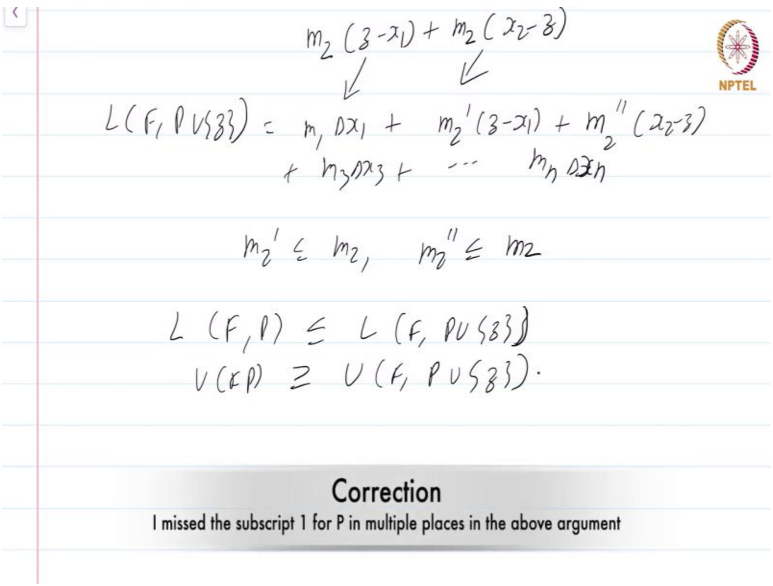

So, essentially what we have shown is $L(f,P) \leq L(f,P \cup [z\})$, ok. Similarly, $U(f,P) \geq U(f,P \cup \{z\})$; exact analogous argument will hold. Why you get $U(f,P)$ greater than or equal to? Is simply because when you take the maximum on a smaller interval, it can only decrease ok; not the maximum, the supremum.

When you take the supremum on a smaller interval, it can only decrease. So, we have these two inequalities; so we have essentially proved this particular claim that when you consider a refinement, the upper sums will have to decrease and the lower sums will have to increase.

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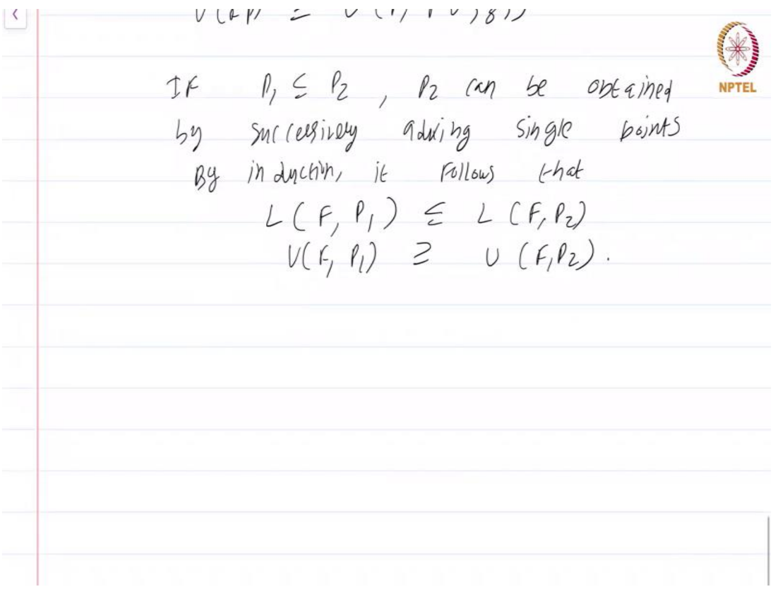

We have just proved it for one point but if $P_1 \subseteq P_2, P_2$ can be obtained; can be obtained by successively adding single points, right. You keep adding one point at a time by induction it follows that $L(f, P_1) \leq L(f, P_2)$ and $U(f, P_1) \geq U(f, P_2)$.

Induction; an easy argument through induction finishes this proof that the lower sums will have to increase and the upper sums will have to decrease, when you do a refinement. Now, we might have been derailed from our original track; our original track was to prove that $L(f, P) \leq U(f, Q)$. Well, that is easy from what we have shown now.

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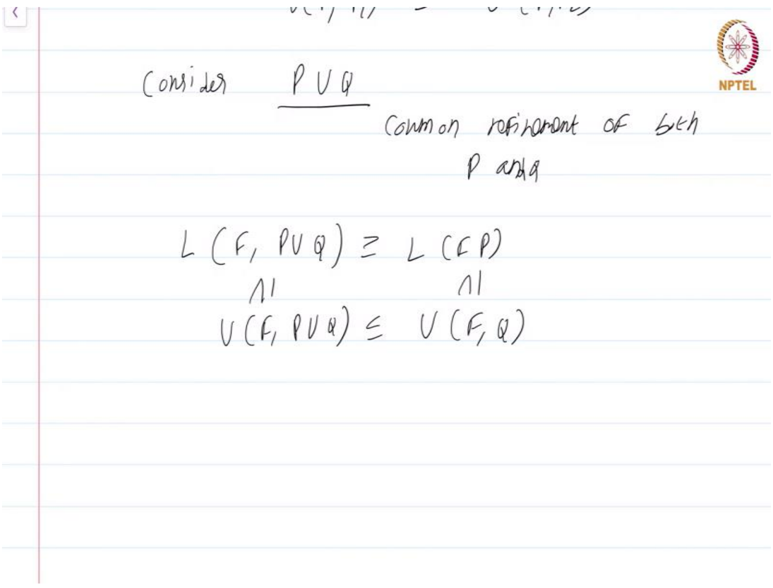

Consider $P \cup Q$; so this is a common refinement of both P and Q right. So, this is a partition that contains both P, as well as Q; so it sort of a common refinement of both P and Q. So, how does this help us? Well, we know that $L(f, P \cup Q) \geq L(f, P)$, right. And this $L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$.

So, this chain shows that $L(f, P) \leq U(f, Q)$, which is what we wanted to prove right. So, essentially this argument was we know the behavior of each one of these sums $L(f, P)$ and $U(f, Q)$ when you refine a partition. But the refinement allows us to compare $L(f, P)$'s and $L(f, P \cup Q)$ and $U(f, P \cup Q), U(f, Q)$ combining all these inequalities together, we get a proof ok.

So, what this suggests is as you keep adding more and more points. Let us go back to the original picture, as you keep adding more and more points what is intuitively happening is that these regions that are enclosed within the area under the graph; these small rectangles that we have used they will become larger and larger whereas, the rectangles on the outside will become smaller and smaller right.

So, this is sort of mimicking what was there in the final axiom of the axiomatic characterization of area. So, you expect that when you add more and more points, these approximations get better and better.

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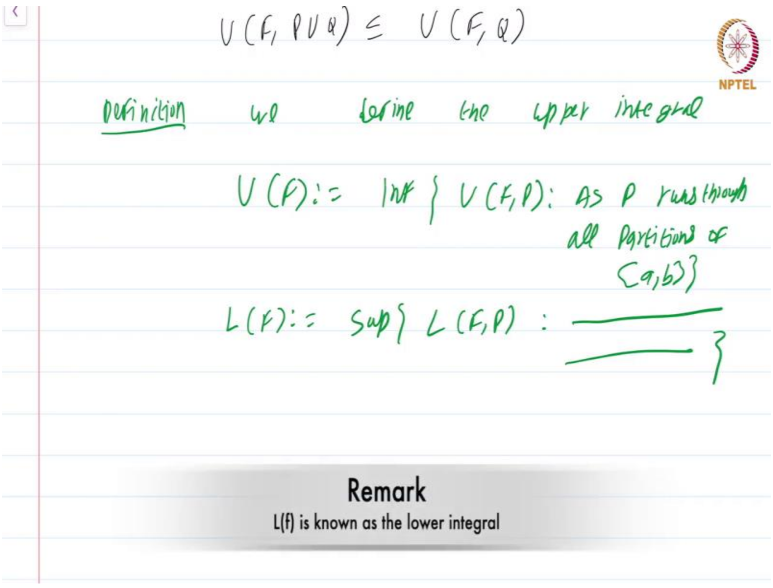

So, now the next definition should not be a shocker at all. Definition: We define the upper integral $U(f) = \inf\{ U(f,P)$:P runs through all partitions [a, b] and similarly $L(f) = \sup\{ L(f,P)$:P runs through all partitions [a, b], ok. Now, when do you say the function f is integrable?

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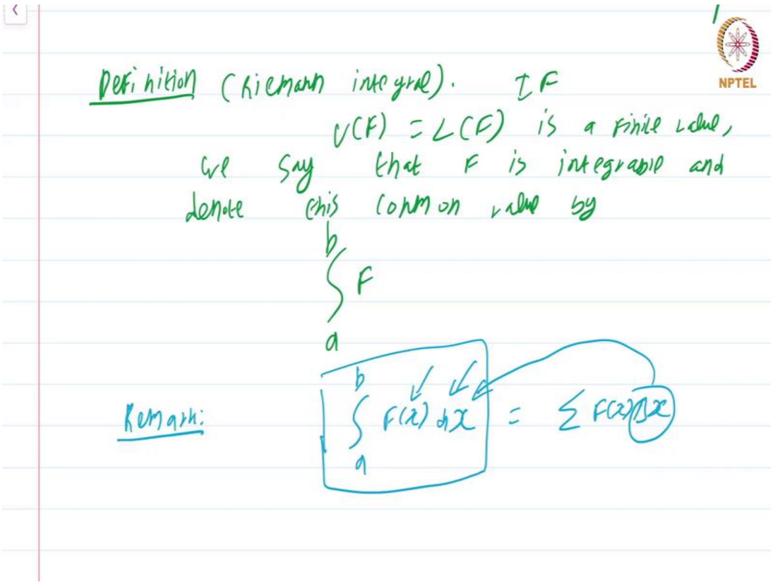

Well, let us highlight it as another definition, the definition of the Riemann integral. If $U(f) = L(f)$ is a finite value, we say that f is integrable and denote this common by $\int_a^b f$, ok. Now, let me end this module with just a remark.

Remark; you will sometimes encounter the notation integral $\int_a^b f(x)dx$. Now, this is a notation, that I neither fully understand what the motivation was nor do I see how it adds clarity to the situation. You are introducing a variable x and you are putting a dx; presumably what was trying to capture is its analogy with some $f(x)$; Δx and you are just sort of shrinking Δx to 0; so, you want to represent that infinitesimal quantity by dx.

Now, I guess this was the original motivation, but to actually make sense of this precisely; why you can write this as $\int_a^b f(x)dx$, requires quite a lot of machinery, the modern tools called differential forms to make this precise. So, I will try to avoid this notation of writing the integral as $\int_a^b f(x)dx$ ok.

According to me, it really does not add any clarity. However, this notation is prevalent throughout the literature and as with many things you cannot change centuries old convention. My favorite example is that of the electron, why would you call the charge of the electron negative?

The correct convention should be that the electron should be positively charged; after all, all of electricity is just movement of electrons, not of some positively charged quantities, it is of negatively charged quantities; now, in our current unfortunate convention. We cannot go back in time and make the charge of electron positive; we have to live with it.

Similarly, we cannot avoid this notation of $\int_a^b f(x)dx$; many times, writing it like this might help in computation, when you are thinking about it very intuitively and informally, but it is sort of misleads us when you are doing things formally and rigorously ok.

So, this concludes the definition of the Riemann integral. In the next few modules, let us see how we determine whether a function is Riemann integrable. Whether the Riemann integrable satisfies those two properties that we had listed as a characterization of the integral; if it does not then this whole exercise would have been futile. So, we will see all that in the coming modules.

This is a course on real analysis and you have just watched the module on the definition of the Riemann integral.