Real Analysis - I

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Lecture – 25.2
Proof: axiomatic Characterization

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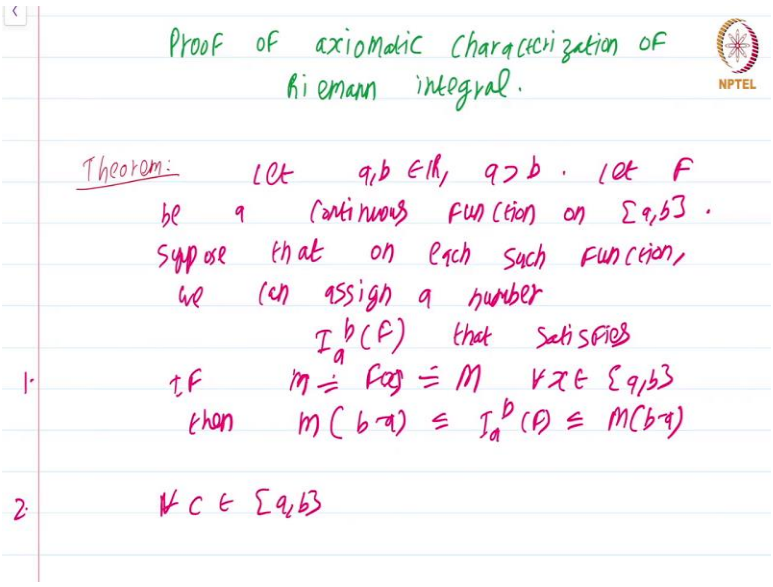

Let me begin by stating the axiomatic characterization of the Riemann integral again, before we go to the proof. Theorem, theorem is as follows. Let $a, b \in R$ $a > b$, let f be continuous function on closed interval $[a, b]$.

Suppose, that on each such function each such function we can assign we can assign a number, $I_a^b(f)$ that satisfies, (1) If small $m \leq f(x) \leq M$, for all $x \in [a, b]$. Then then $m(b-a) \leq I_a^b(f) \leq M(b-a)$.

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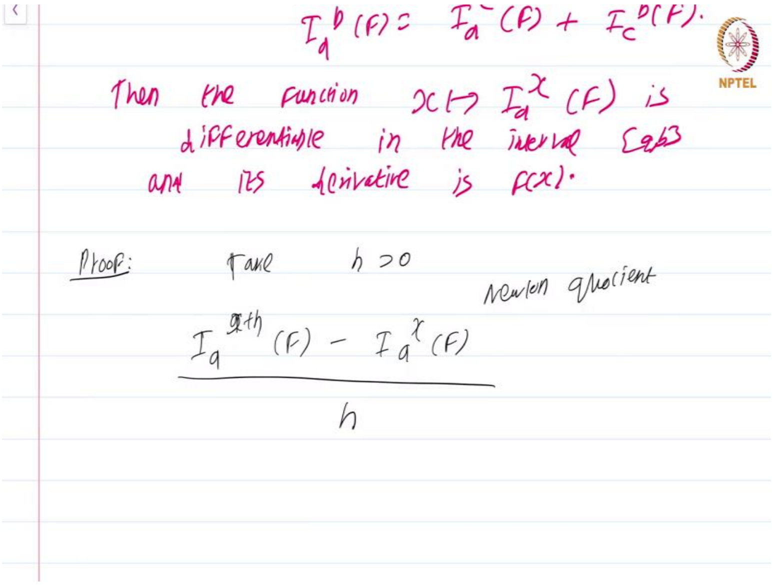

And property 2 is for all $c \in [a, b]$; $I_a^b(f) = I_a^c(f) + I_c^b(f)$, ok. Needless to say, this function I_a^b is defined on all continuous functions defined on closed interval $[ab]$. I am assuming that given any closed interval, there is an associated I function that sort of is supposed to measure the area ok.

So, the way I have written you might be misled into thinking that this I_a^b function is defined, specifically for this particular pair a, b, but that is not the case. for each continuous function on any closed interval, there is an associated function.

Then the function $x \mapsto I_a^x(f)$ is differentiable, in the interval in the interval (a, b) and its derivative and its derivative is small $f(x)$. So, if you differentiate the integral you are supposed to get back the function that is what this is saying, though we do not really know what this $I_a^b(f)$ is so far, it is just something that is assumed to exist and satisfying these two properties.

The proof for all the buildup with the statement is actually quite easy proof. So, for instance what we will do is, we will first take, take $h > 0$; the $h < 0$ case is exactly similar, word for word you just have to change a few signs and everything will drop out in your lap, I leave it to you.

What we have to do is find $\frac{I_a^{x+h}(f) - I_a^x(f)}{h}$, this is the Newton quotient. We are going to compute the Newton quotient by hand ok. Now, here is where things get a bit helpful, because we have properties that we need.

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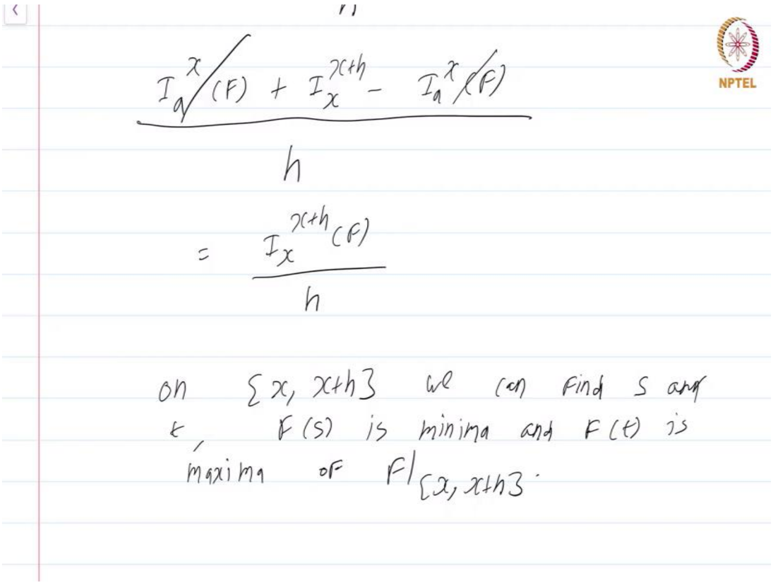

We can write this as $\frac{I_a^x(f)+I_x^{x+h}-I_a^x(f)}{h}$, this is just by property 2 of the function, I_a^b or I_a^x here. And immediately we see that something nice happens, we are left with $\frac{I_x^{x+h}(f)}{h}$, ok. Now, on $[x, x+h]$, you are understanding why I have taken $h > 0$, now, so that proof becomes easier to write down exactly similar arguments will hold when $h < 0$.

We can find, let us say s and t such that $f(s)$ is minima and $f(t)$ is maxima of $f|_{[x.x+h]}$. On any closed interval, continuous functions attain maxima and minima, I am just choosing the corresponding points to be s and t ok.

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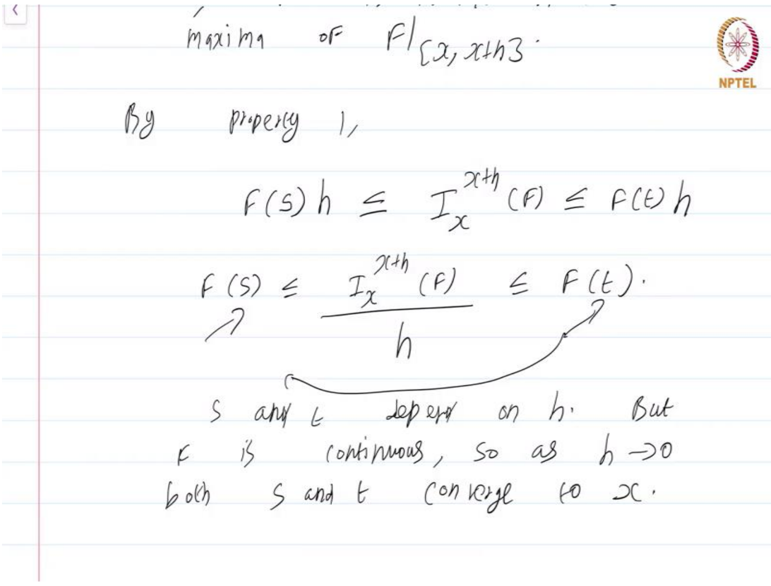

So, what is this mean we know that by property 1, by property 1 we know that m or rather $f(s)h \leq I_x^{x+h}(f) \leq f(t)h$, this is just property 2; sorry, this is just property 1 ok. Now, now that means $f(s) \leq \frac{I_x^{x+h}(f)}{\text{h}} \leq f(t)$, ok.

Now, note that, you might be deceived into thinking that this s and t are constant, but technically s and t, depend on h, but f is continuous. So, as h goes to 0 both s and t, converge to the point x right, so this is saying nothing essentially you have x, $x + h$ ok.

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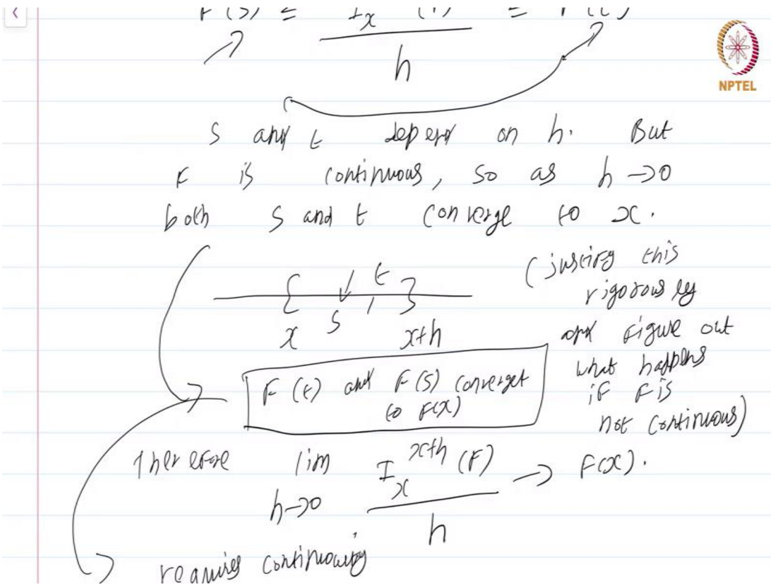

You have chosen the points s and t somewhere in this interval ok, as $x + h$ converges to x; these points s and t will have to go to x and this just follows by continuity of the function f, this is the place where we have used the continuity crucially ok. Now, I want you to justify this rigorously and figure out what happens if f is not continuous, figure out what happens what happens if f is not continuous, ok.

Therefore, $\lim_{h \to 0} \frac{I_x^{x+h}(f)}{h} = f(x)$, because it is sort of squeezed between $f(s)$ and $f(t)$. And both $f(s)$ and $f(t)$ sort of approach x both s and t, wait a second I might have misled you a bit both s and t converge to x that is not that is not dependent on continuity that just follows, because s and t has squeezed between x and $x + h$. What is to be; what is actually continuity is going to say is that $f(s)$ and $f(t)$ converge to $f(x)$ ok, this is the part that requires continuity this is the part that requires continuity this is the part that requires continuity.

So, let me just erase; but as h converges to 0 both s and t converged x. Therefore, because f is continuous $f(s)$ and $f(t)$ both converge to $f(x)$, this is the part I want you to this is the part I want you to rigorously justify, this is actually just one line ok. So, what requires continuity is the fact that $f(s)$ and $f(t)$ both converge to $f(x)$, and this delivers the proof this is the proof that is all the proof is so simple this is the proof, ok.

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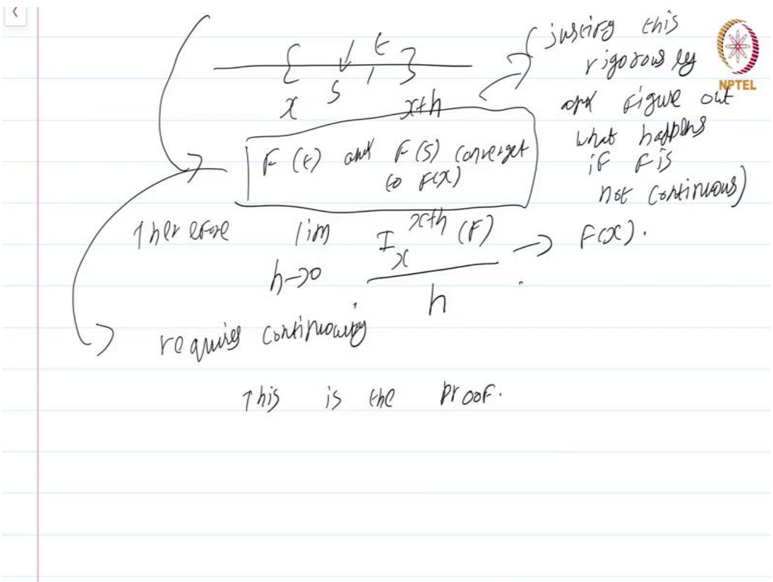

So, we have now gotten a proof that this function I which is modeled on the area function, but does not seem to satisfy all the requirements that we listed out of area; just seems to satisfy some basic requirements like this additivity which is very basic. And the fact that if you can squeeze in a smaller rectangle and a larger rectangle inside and outside the region.

Then the area of the rectangles you should have this relationship; these are obvious things that you require of area. Just with these requirements you have something like the fundamental theorem of calculus, ok. Now, let us try to geometrically interpret, what this is trying to say ok.

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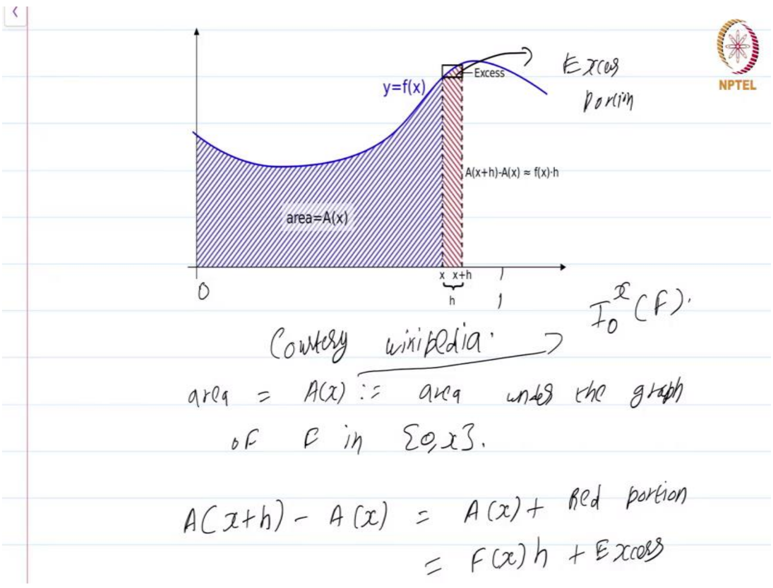

So, what I will do is, I am going to borrow this figure from Wikipedia, so this is courtesy Wikipedia ok, I am just borrowing this figure and let us try to see what it is trying to say ok. So, you have this graph of this continuous function $y = f(x)$ ok. And for each point x, so this is going from, let us say it starting at 0 and going all the way somewhere, somewhere let us say till 1, ok.

Now, what we are assuming is that you have this function area equal to define to be $A(x)$; which is supposed to be area under the graph, under the graph of f in $[0, x]$, ok. So, you have a function that measures the area in our notation this is supposed to be captured, but we have not yet even constructed it, but it supposed to be captured by a $I_0^x(f)$; this is essentially the function we considered. So, we are assuming we have an area function. This function I will turn out to be the area, which we will see in the construction in the next lecture.

Now, let us see what happens why geometrically you would expect this area function to be related to the derivative ok. Now, notice that if you want to compute $A(x + h) – A(x)$ which is what we were interested in, what you can do is you can just take it as area of x, approximately equal to I will use this tilde notation approximately equal to $A(x) +$ red portion; that red portion right. So,$A(x + h) = A(x) + area\ of$ red portion in this picture. Now, that red portion area this is not approximately equal to, this is exactly equal to this is exactly equal to.

Now, this red portion is approximately equal to $f(x)h$ right, this is just $f(x)h$. In fact, we can write it as $f(x)h + excess\ portion$, what they have called excess this part, this part this is the excess portion fine. So, we have written down $A(x + h) - A(x)$ as $f(x)h + excessportion$ ok.

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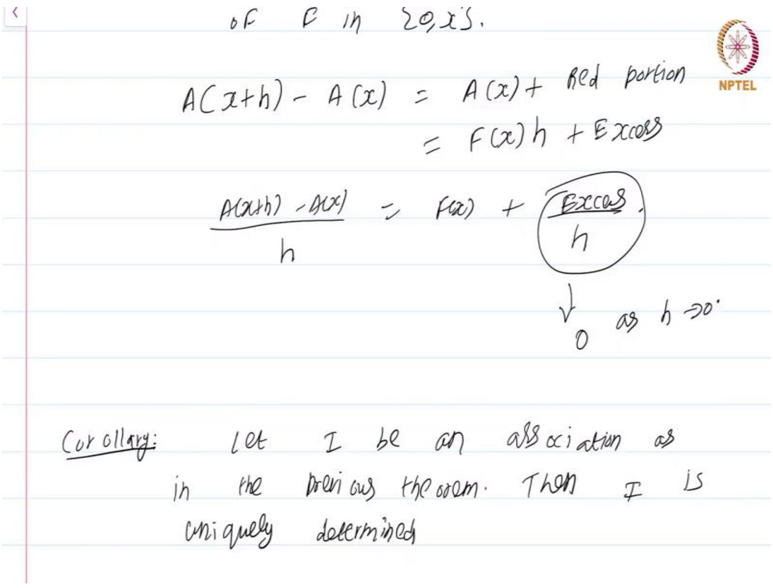

Now, it is really easy to see why areas is related to the derivative we all we are doing is we are taking $\frac{A(x+h)-A(x)}{h} = f(x) + \frac{excess}{h}$. And what is really happening is by continuity this goes to 0, this goes to 0 as h goes to 0 right that is essentially that is essentially the entirety of the proof ok.

And the rigorous justification that the excess portion does indeed excess portion divided by h does indeed, go to 0 is what we have just presented, except the I function that we have we do not know whether it is the area, it is just a function that has two basic properties ok.

Now, why would you expect this I function whose existence itself is yet to be shown is actually going to measure the area, since it satisfies only two basic requirements of area not all. Well, this next corollary with this geometric interpretation that we have should convince you. If it does not, do not worry; in the next lecture, when I am going to rigorously construct this I function using Riemann sums and upper sum, lower sum, and so on. It will become geometrically obvious that in fact yes, this I function is in fact going to measure the area ok.

Let I be the association be an association that is a better way to put it, be an association. Note, there could be many many functions so far, this all I am saying is given any closed interval and a continuous function on that closed interval there is some way to assign a number $I_a^b(f)$. There could be more than one way that satisfies conditions 1 and 2 that we have stated in the previous theorem that I be in an association, as in the previous theorem.

Then I is uniquely determined, there is only one such function. So, from combining with the previous geometrical argument we have to note that if at all there is an I function that exists, it is got of got to measure the area this is more made more precise in the second part.

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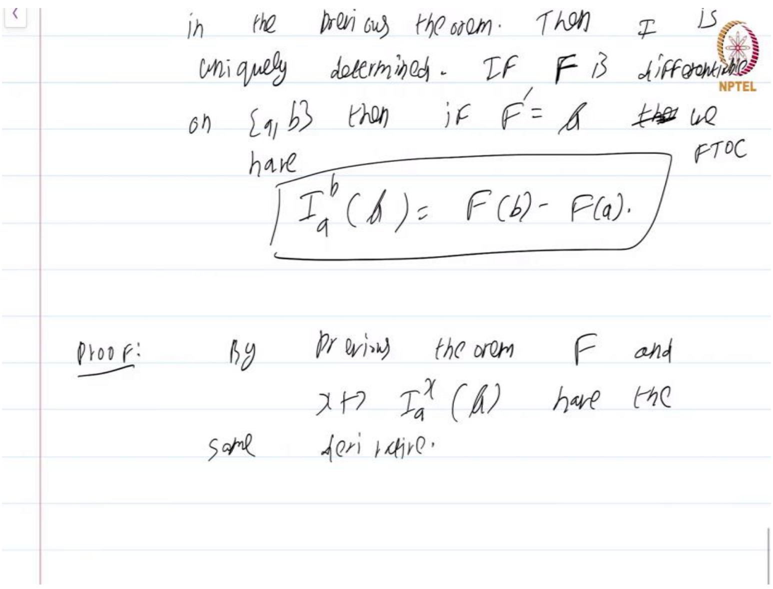

If capital F is differentiable on closed interval $[a, b]$; then, if capital $F' = f$, we have, as you can expect, $I_a^b(f) = F(b) - F(a)$ ok. So, what we get is the following if at all there is an association I, then I_a^b(f)= $F(a) - F(b)$, it must sort of satisfy the fundamental theorem of calculus. This is the fundamental theorem of calculus, I mean the conclusions of the previous part and the conclusions of this part are sort of two versions of the fundamental theorem of calculus.

And if you think about it with this geometrical interpretation that we just saw, it will sort of force this function $I_a^b(f)$ to be the area under the graph function ok. So, I this is something that you have to realize for yourself; me coming and explaining for 15 minutes repeatedly was not really going to make an impact.

I urge you to sit down look through this geometric interpretation, look through the proof of the previous theorem and the proof of this corollary which is just two lines proof. Now, by previous theorem; $F(x) = I_a^x(f)$ have the same derivative that is what we showed, have the same derivative ok.

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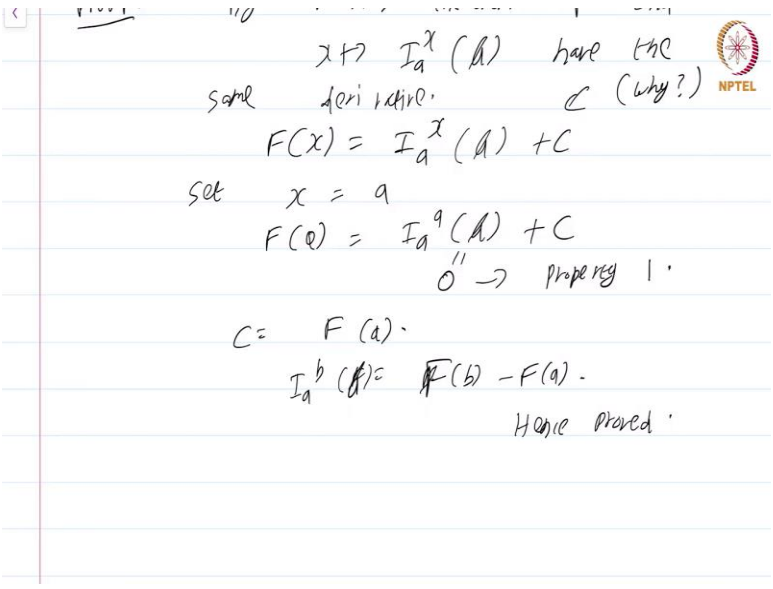

Now, if two functions have the same derivative, then you can show that $F(x) = I_a^x(f) + C$, why? This is just one line from a rather important theorem that we saw in the chapter on derivatives, it is going to be rather easy to show this ok.

So, $F(x) = I_a^x(f) + c$; now set x = a ok, so you will get, $F(a) = I_a^a(f) + c$, but $I_a^a(f) = 0$ this is by property 1, this is by property 1 by; property 1 $I_a^a(f) = 0$.

So, we get $c = f(a)$ we get $c = f(a)$. So, in other words $I_a^b(f) = F(a)$, so this should be small f. $I_a^b(f) = F(b) - F(a)$hence proved ok.

So, this is really interesting, this is really interesting. We have now shown, we have now shown this function $I_a^x(f)$ whose existence is still up in the air seems to be the candidate for the area under the graph function. So, our goal in the next lecture is to give a proper construction of this function $I_a^x(f)$.

This is a course on Real Analysis. And you have just watched the module on the Proof of the Axiomatic Characterization of the Riemann Integral.