Real Analysis - I
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Lecture - 24.2
The Ratio Mean Value Theorem and L_Hospital_S Rule

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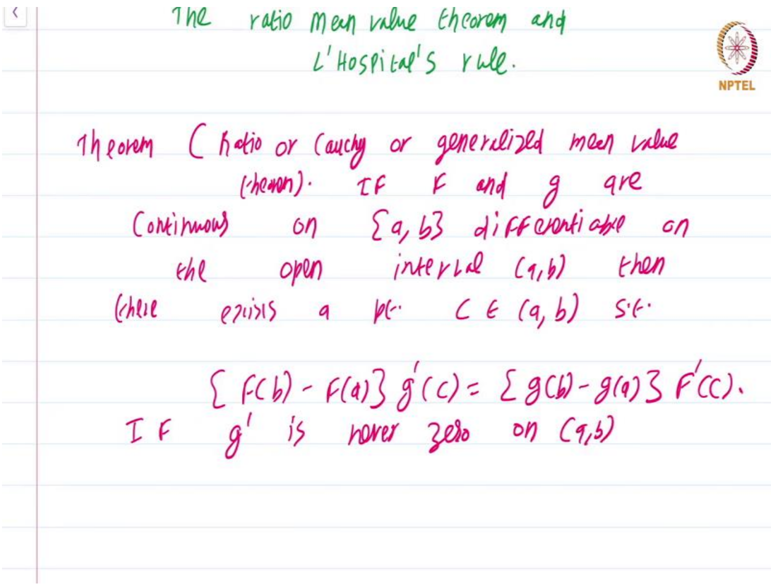

In this module, we shall first see a generalization of Lagrange's mean value theorem commonly known as the Ratio Mean Value Theorem or the Cauchy mean value theorem or the generalized mean value theorem. And then we shall apply it to prove the famous L' Hospital's rule that is applied right and left by JE students without actually knowing what the statement is.

So, let us begin with the statement and proof of the ratio mean value theorem, ratio or Cauchy or generalized mean value theorem. So, the statement is as follows, instead of one function we now have two functions. If f and g are continuous on the closed interval $[a, b]$ differentiable on the open interval (a, b); then there exists a point $c \in (a, b)$ such that $f(b) - f(a)$.

So far this looks very familiar, but $f(b) - f(a)g'(c)$ is equal to as you can guess $g(b) - g(a)f'(c)$. Now, it might be perplexing why this is called the ratio mean value theorem.

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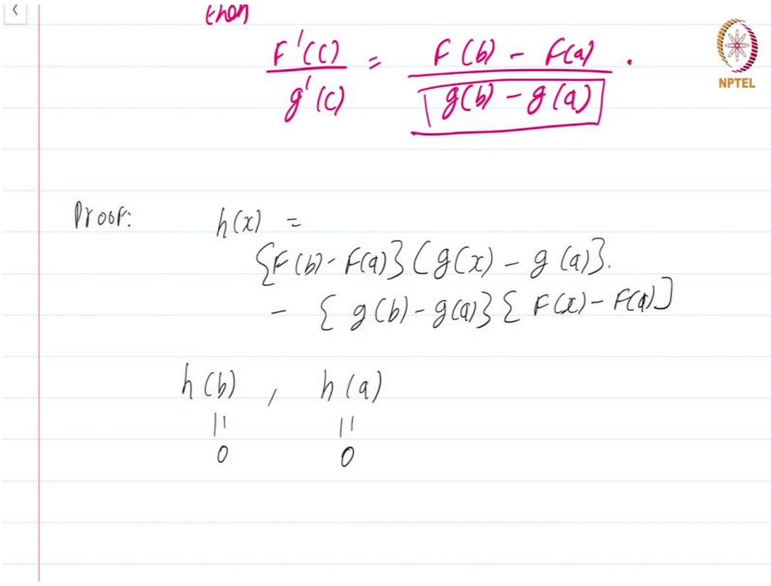

Well, if $g' \neq 0$ on open interval (a, b), then of course, I can write it as ratios; $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$, fine. Now, this is the generalized mean value theorem simply because if you take the function g to be the function $g(x) = x$, the identity function, you recover the Lagrange's mean value theorem.

Another comment to be made the eagle-eyed reader might be wondering why this is not 0. Well, it is not 0 because we have assumed $g' \neq 0$ on (a, b). Think about why $g' \neq 0$ will force $g(b) \neq g(a)$, ok.

So, let us prove this. The proof is not really hard. Proof: Well, as you can guess we are going to apply Lagrange's mean value theorem to a special function. Well, we just choose $h(x) = \big(f(b) - f(a)\big)\big(g(x) - g(a)\big) - \big(g(b) - g(a)\big)\big(f(x) - f(a)\big)$ ok. So, we are just going to apply Lagrange's mean value theorem to this function.

first let us check what $h(b)$ is and what $h(a)$ is. Well, by the way things have been chosen when I substitute $x = b$ everything gets cancelled and you get 0. And when you substitute $x = a$, also you get 0 that is the way these functions have been chosen.

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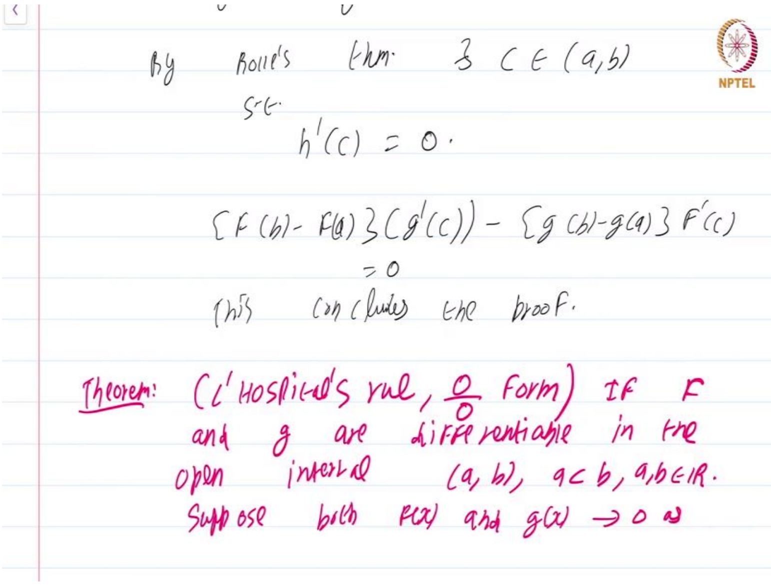

So, you do not even know, you do not even need to apply the full force of Lagrange's mean value theorem just by Rolle's theorem. Just by Rolle's theorem, there exists some $c \in (a, b)$ such that such that $h'(c) = 0$ right. Now, you can immediately see that $h'(c) = \big(f(b) - f(a)\big)g'(c) - \big(g(b) - g(a)\big)f'(c) = 0$, and this concludes the proof.

The second part where I am assuming that $g' \neq 0$ everywhere follows immediately ok. So, now, we are going to apply this ratio mean value theorem to prove one version of L' Hospital's rule. This is not the only version, there are several versions, please check the notes. So, theorem, L' Hospital's rule, L' Hospital's rule $\frac{0}{0}$ form: just writing $\frac{0}{0}$ makes my hand curl up in agony, but that is the way this is usually stated.

If f and g, f and g are differentiable in the open interval (a, b); so, I am just going to take this to be a finite open interval; suppose, both $f(x)$ and $g(x)$ converge to 0 as x approaches b ok.

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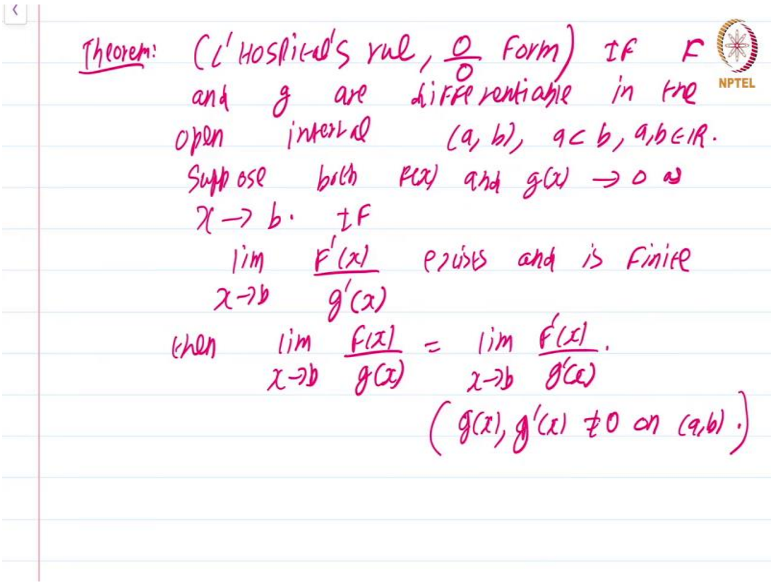

If $\lim_{x \to b} \frac{f'(x)}{g'(x)}$ exists and is finite and is finite, then $\lim_{x \to b} \frac{f(x)}{g(x)} = \lim_{x \to b} \frac{f'(x)}{g'(x)}$ ok. So, what we are essentially assuming for all these to go through is that $g(x)$ and $g'(x)$ are never 0 on (a, b) ok, that is being assumed here, otherwise the quotients would not even make sense.

So, the statement says that if you have two functions that are approaching 0 with the denominator never being 0, then the limit of the ratio $\frac{f(x)}{g(x)}$ which is going to approach $\frac{0}{0}$ form will actually exist if $\lim_{x \to b} \frac{f'(x)}{g'(x)}$ exists, and not only will the limit exist, it will coincide with limit $\lim_{x \to b} \frac{f(x)}{g(x)}$, ok.

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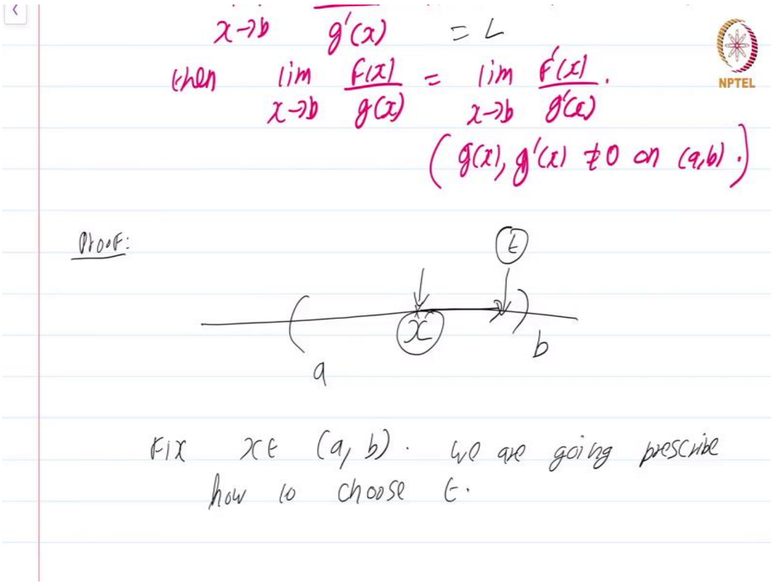

Now, here is one scenario where it is better to use the terminology and language that we introduce to speak about limits such as some quantities being small, as you up get close enough to other quantities so on right. So, we had introduced this notion of as you get arbitrarily close to and things become arbitrarily small and so on.

By wording this proof in that language, the proof becomes really transparent and clear. Whereas, when you write down the nitty-gritties in terms of epsilon and delta the proof becomes really convoluted. So, what I am going to do is, I am going to just write the proof using the full force of language. Language after all is just a shortcut that eases the process of thinking.

So, I am going to exploit the full range of our vocabulary in this proof, but nevertheless this is an introductory course getting too comfortable with language might hide the difficulties and make you convince yourself that you have understood when in reality you are just deluding yourself.

So, what I have urge you to do is to read this proof, process it, understand it and translate it to rigorous mathematics ok. Not that the proof I am about to give is not rigorous it is just using a lot of shortcuts ok. Now, what is the idea behind the proof? Well, what we are going to do is just look at the interval (a, b). We are approaching we are approaching the point b right.

Now, what I am going to do is, suppose the point x is here, I have to show that $\lim_{x \to b} \frac{f(x)}{g(x)} = \mathrm{L}$ just call this is call this L; $\lim_{x \to b} \frac{f'(x)}{g'(x)}$, call this L; I want to show that this ratio also converges to L. What I do is, given this x which is close to b. I choose another t which is much closer to b ok, I choose this t this t will of course, depend on x; I choose this t much closer to b.

Now, when you are very, very close to b, we know that $f(t)$ and $g(t)$ are going to be exceptionally small. In fact, by moving this t, sufficiently close to b we can make a both $f(t)$ and $g(t)$ as close to 0 as we desire ok.

And then what I am going to do is I am going to apply the ratio mean value theorem to the functions f and g between these points t and x. So, given any x, I will always be able to find a t such that certain nice things happen and that is essentially going to be the proof ok.

So, fix $x \in (a, b)$ ok, we are going to choose, we are going to prescribe, rather prescribe how to choose t, how to choose t ok. Now, what is the ratio mean value theorem say we are assuming that both $g(x)$ and $g'(x)$ are never 0.

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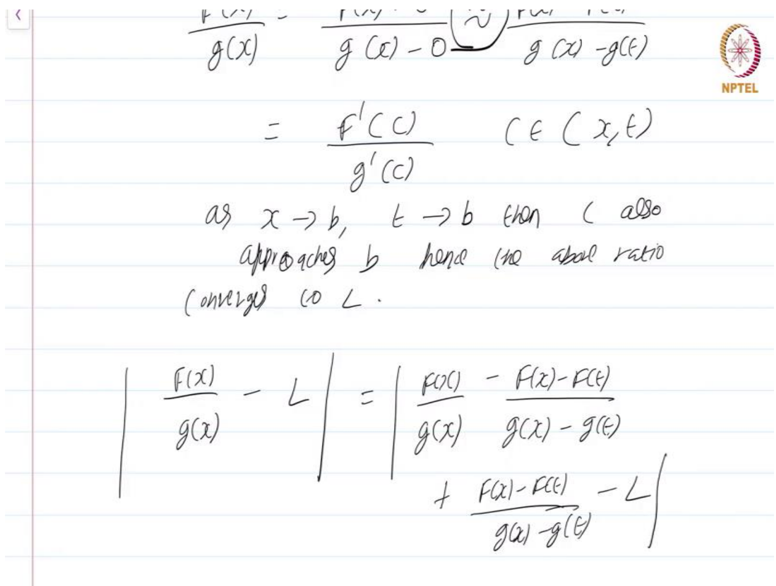

So, what we can do is, this $\frac{f(x)}{g(x)}$, we can write it as $\frac{f(x)-0}{g(x)-0}$ correct. We can do this. And this is approximately equal to f$\frac{f(x)-f(t)}{g(x)-g(t)}$, that this approximation can be made as nice an approximation as you desire by choosing t appropriately correct.

Now, because of this, we can write this as we can write this as $\frac{f(c)}{g(c)}$, where c lies in (x, t), where this lies in (x, t), ok. Now, now this as x approaches b, t also approaches b we have just given an x we can always choose t very very close to b, then c also approaches b; because c is squeezed between x and t ok. Hence, the above ratio converges to L, ratio converges to L. Therefore, the original $\frac{f(x)}{g(x)}$ also converges to L.

Now, this is a very rough sketch very rough sketch that illustrates the idea behind the proof. So, let us make it somewhat better. This is a little bit too vague. Let us make it somewhat better and add some details ok. So, we essentially get the idea how to choose this t, you choose this t very very close to b, so that this approximation becomes very close to an identity.

So, what is it that we want to do? we want to analyze $\frac{f(x)}{g(x)} - L$, ok. We want to analyze this. Now, what we are going to do is, we are going to add and subtract terms involving $f(t)$;

$\left|\frac{f(x)}{g(x)}\right| - L = \left| \frac{f(x)}{g(x)} - \frac{f(x)-f(t)}{g(x)-f(t)} + \frac{f(x)-f(t)}{g(x)-f(t)} - L\right|$ f, ok. We want to make this quantity arbitrarily small if x is sufficiently close to b.

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$$\le \left| \frac{f(x)}{g(x)} - \frac{f(x)-f(t)}{g(x)-g(t)} \right| + \left| \frac{f(x)-f(t)}{g(x)-g(t)} - L \right|$$

$$= \left| \frac{f(x)}{g(x)} - \frac{f(x)-f(t)}{g(x)-g(t)} \right| + \left| \frac{f'(c)}{g'(c)} - L \right|$$

Fix $\varepsilon > 0$ $\quad$ $c \in (x, t)$.

Now choose t so that, the first term is $< \frac{\varepsilon}{2}$

Now, by triangle inequality of course, we get $\left|\frac{f(x)}{g(x)} - \frac{f(x)-f(t)}{g(x)-f(t)}\right| + \left|\frac{f(x)-f(t)}{g(x)-f(t)} - L\right|$, fine. And again this is equal to get $\left|\frac{f(x)}{g(x)} - \frac{f(x)-f(t)}{g(x)-f(t)}\right| + \left|\frac{f'(c)}{g'(c)} - L\right|$, where c lies between (x, t) where c lies between (x, t).

Now, choose t so that this term the first term the first term first term is less than $\frac{\varepsilon}{2}$. So, fix $\varepsilon > 0$. Choose t, so that the first term is less than $\frac{\varepsilon}{2}$. Now, we have choice of this t, and this choice of t depends on x; I have not told you how to choose x, ok only then can we proceed to choose the point t actually.

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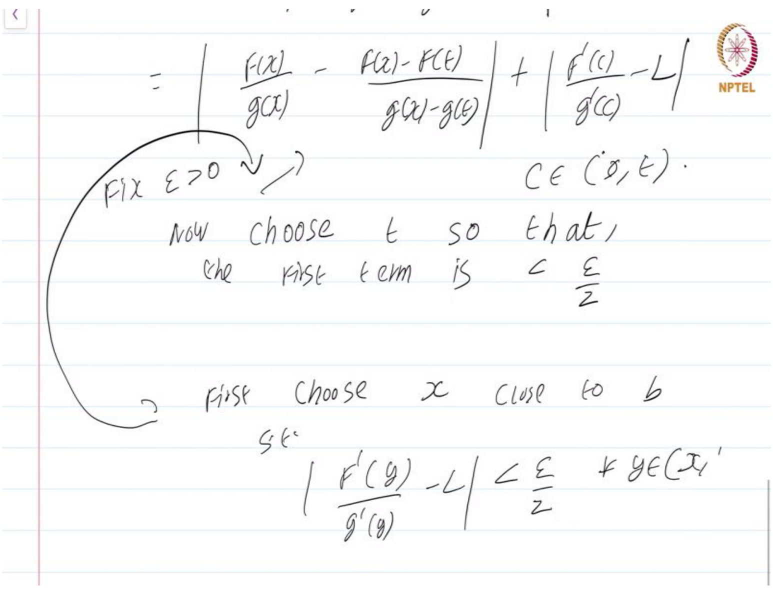

So, this should actually come first this should actually come first. So, let me put the arrow this way. first choose, x close to b such that $\left|\frac{f'(c)}{g'(c)} - L\right| < \frac{\varepsilon}{2}$ for all $y \in (x, b)$. So, just choose this point x sufficiently close to b, so that the ratio of the derivatives is already less than $\frac{\varepsilon}{2}$ when you take the difference with the limit L ok.

So, first choose this x, so that this happens. Now, choose t sufficiently close to b, so that this term also becomes less than $\frac{\varepsilon}{2}$, ok.

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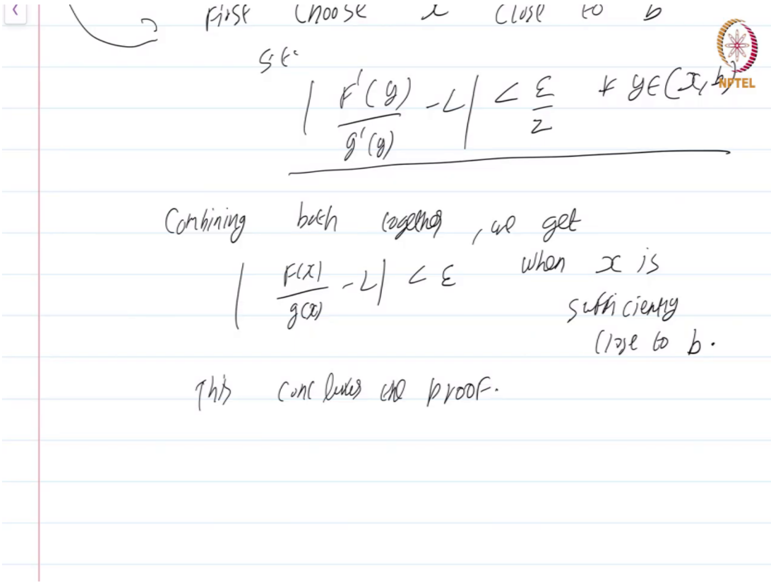

Combining both together, combining both together combining both together both together, we get $|\frac{f(x)}{g(x)} - L| < \varepsilon$ when x is sufficiently close to b sufficiently close to b ok. And this concludes the proof, this concludes the proof, concludes the proof ok.

So, the only step that is actually vague is the choice of this point x is not really vague. I am just saying that there will be some there will be for sufficiently close to b, there will be you can choose such that $|\frac{f'(y)}{g'(y)} - L| < \frac{\varepsilon}{2}$, that is just coming from the definition of limit. So, there is no impreciseness here. The impreciseness is to say that this quantity can be made less than $\frac{\varepsilon}{2}$ if t is sufficiently close to b.

I leave it to you to determine in terms of the functions f and g, how you have to choose this point t that is not really hard. But it is clear that it can be done you just have to expand this $\frac{f(x)}{g(x)}$ - $\frac{f(x)-f(t)}{g(x)-f(t)}$, you just have to expand it and analyze those terms and determine how you have to choose this point t ok. So, I leave it to you to complete the proof. So, this concludes this module.

You are watching the course on Real Analysis, and you have just watched the module on the Ratio Mean Value Theorem and L' Hospital's rule.