Real Analysis - I
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Lecture – 23.3
Applications of the Mean Value Theorem

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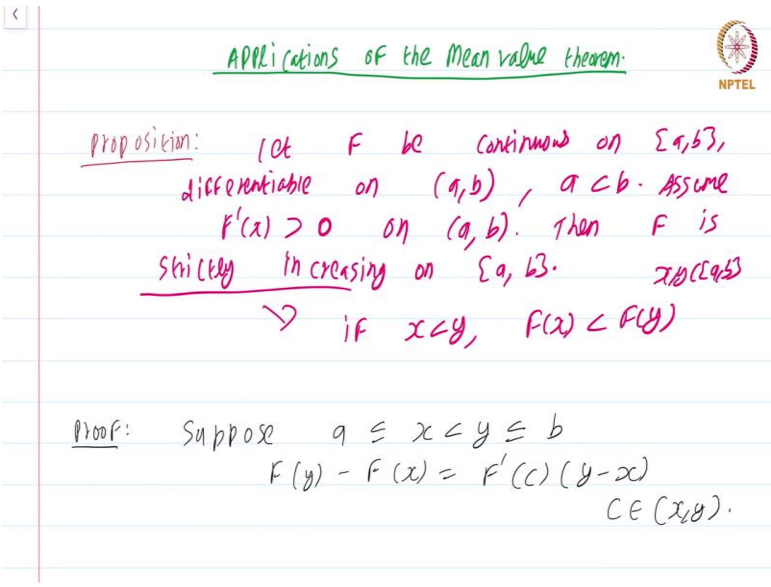

Let us see some nice Applications of the Mean Value Theorem. I will first state one property of the derivative that you are no doubt familiar with. The design of the derivative determines whether the function is increasing or decreasing and you can use the sign to predict it. Let us make this into a rigorous statement;

Proposition, let f be continuous on the interval $[a, b]$, differentiable on open interval (a, b) and of course, $a < b$. Assume the derivative $f'(x) > 0$ on (a, b). Then, f is strictly increasing on closed interval [a, b].

To clarify, this strictly increasing just means that if $x < y, f(x) < f(y)$ right; $x, y \in [a, b]$. This is just a clarification of what it means for a function to be strictly increasing.

Similarly, I leave it to you to formulate; what is strictly decreasing, what is just increasing and what is just decreasing and just decreasing will just have $\leq$ here ok. So, let me just erase that $\leq$ so, that there is no confusion.

Proof; Well the statement is long and the proof is I think only maybe even shorter. So, suppose $a \leq x < y \leq b$; choose some x, y in the closed interval [a, b]. Then, $f(y) - f(x) = f'(c)(y - x)$, where this c is coming from open interval (x, y) right. This is just what the mean value theorem says; this is just the mean value theorem ok.

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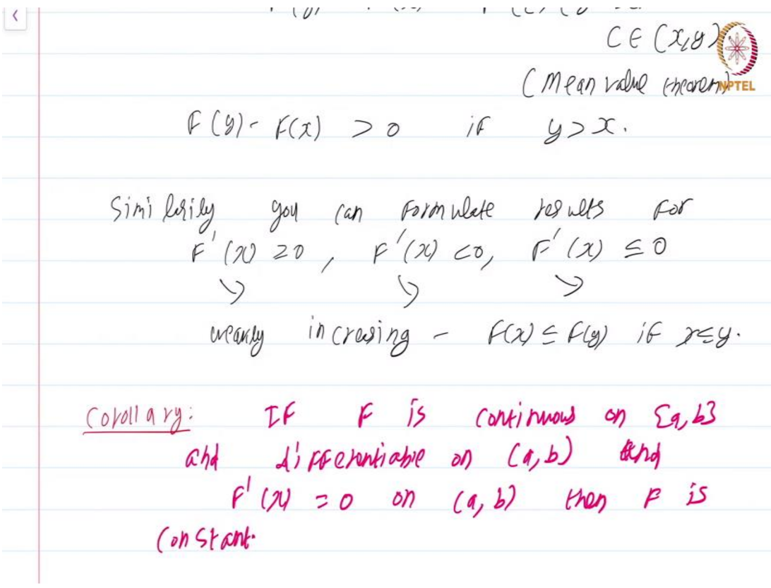

But, that just means $f(y) - f(x) > 0$; if $y > x$ right, simply because $f'(c)$ is always greater than 0 ok. So, I should write greater than 0 because $f'(c) > 0$, you immediately get the result ok. So, similarly you can show, you can formulate results for $f'(x) \geq 0, f'(x) < 0, f'(x) \leq 0$.

Here, you will conclude that f is increasing, here you will conclude that f is strictly decreasing and here you will conclude that f is decreasing ok. Sometimes, to emphasize the distinction between increasing and strictly increasing; we may write weakly increasing ok, that is some terminology used by some people. This just means $f(x) \leq f(y)$; if $x \leq y$.

So, I leave it to you to formulate and prove these easy results; they are nothing new, they are exactly the same. We have a nice corollary of the previous proposition; the corollary says the following. If, f is continuous on closed interval $[a, b]$ and differentiable on open interval (a, b) and if $f'(x) = 0$ on (a, b); then f is constant.

Well, this is a rather easy corollary; once you have established all the results for $f'(x) \geq 0$ and so on, you can apply this proposition that we just proved to both conclude that f is increasing, as well as decreasing which leaves f with the only option of not changing at all ok.

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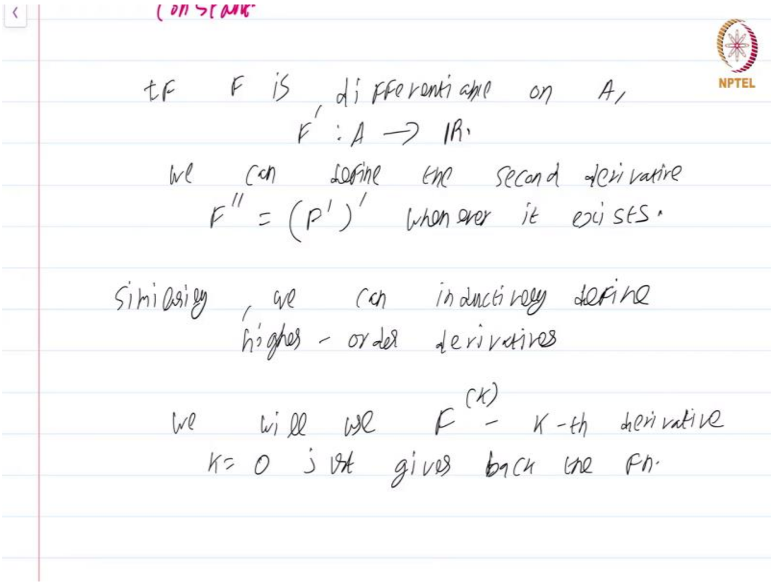

So, this is one interesting set of results that follow immediately from the mean value theorem. Now, I want to introduce some results about the sign of the second derivative. So, what is the second derivative? Well, if f is differentiable on A; we have of course, we can consider this function f': $A \longrightarrow R$ ok; so, this just gives the derivative at every point.

Well, we can define the second derivative, f''; we write it as double prime which is just f'' whenever it exists ok. Note, we get several simple corollaries; if f'' exists at every point of A, then f' is automatically continuous; at every point of A and so on.

Similarly, we can define; we can inductively, that is a better thing to use, inductively define higher order derivatives. Now, this prime notation for derivatives quickly starts to become very very annoying, once you cross 3; even 3 itself looks a bit ugly, I am writing something like $f'''(x)$ ok.

So, better notation is just this; we will use we will use $f^{(k)}$ to denote k times derivative or the kth derivative; that is kth derivative ok. With the additional thing that $k = 0$, just gives back the function just gives back the function; this is just the convenient convention.

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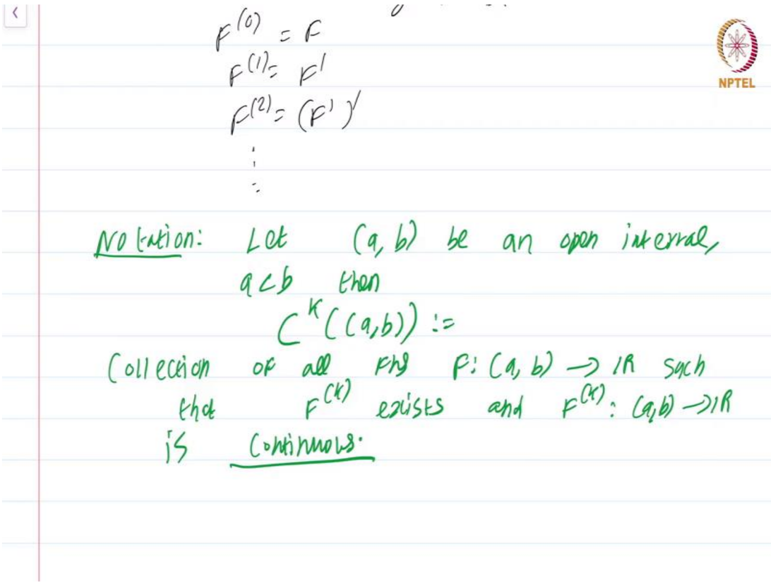

So, $f^0 = f$; $f^1 = f$’; $f^2 = f$” and so on ok. So, the k indicates how many times you are differentiating the function. Now, one important notation; since it is important, I am going to put it in green notation; let (a, b) be an open interval; interval $a < b$, then $C^k(a, b)$; this is defined to be the collection of all functions, let for convenience, let me not put the braces because it is a bit long. Collection of all functions $f: (a, b) \longrightarrow R$ such that $f^{(k)}$ exists and $f^{(k)}$: $(a, b) \longrightarrow R$ is continuous; this is important. So, $C^k(a, b)$ is the collection of all functions $f: (a, b) \longrightarrow R$; such that $f^{(k)}$ exists and $f^{(k)}$ is continuous.

So, if you think about this definition for a couple of moments; you will realize that automatically $f^{(k-1)}$ exists because that is the way $f^{(k)}$ was defined right. for, $f^{(k)}$ to be even defined; $f^{(k-1)}$ must be defined. So, automatically $f^{(1)}, f^{(2)}, f^{(3)}, \dots f^{(k-1)}$ is automatically defined just because I am saying that $f^{(k)}$ exists ok.

So, this $f^{(k)}$exists is automatically asserting that all the lesser derivatives also exist and $f^{(k)}$should be continuous. There is no notation for just functions whose kth derivative exists. To the best of my knowledge there is no notation for that, but there is a notation for this; functions whose kth derivative exists and the kth derivative is continuous.

Just for convenience, I am using open intervals; I do not want to deal with the closed interval case, when I consider such higher order derivatives ok. There is no theoretical issue, you can define it just like that; it will just not come up much in our study.

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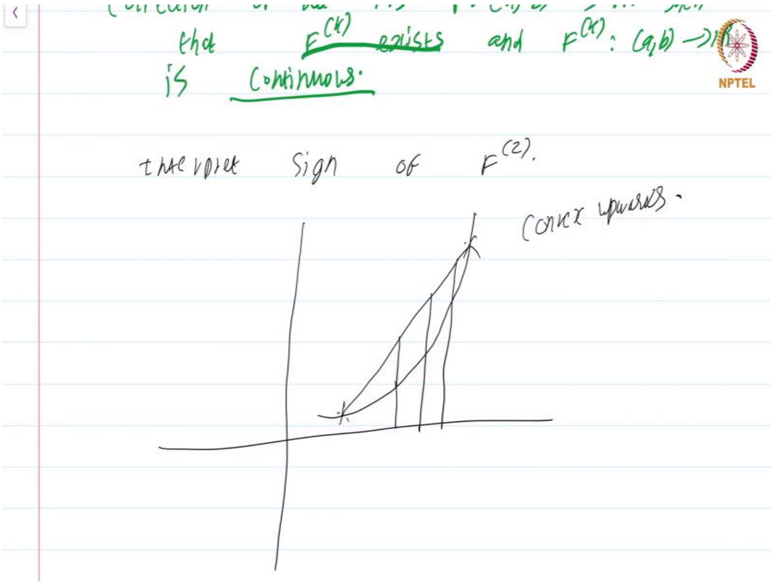

Now, our goal is to interpret sign of $f^{(2)}$, ok. So, for this, first some basic geometry; suppose I have a curve that looks like this, such a curve is called convex upwards. All of you are familiar with convex and concave lenses, this is called convex upwards. This just means that if I take two points on the curve and consider the line joining those two points, then the curve lies beneath; that line always, at all points ok.

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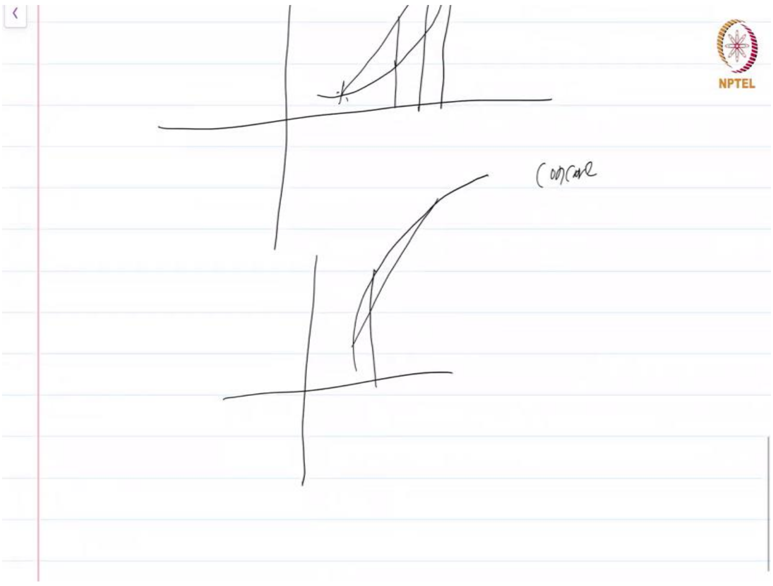

So, the opposite of this is something that looks like this and here if you join two lines, it is not the case that the curve is below the line; that is not happening. Now, you might ask, why do not I just call it convex and call this concave? There is a reason, why we say convex upwards; just think for a moment this notion of concave or convex sort of depends on which direction you are looking from.

A surface that is concave, if you look at it from the opposite direction will become convex and vice versa. So, these are notions that are not defined independent of how the observer is looking. So, it is better to always write convex upwards.

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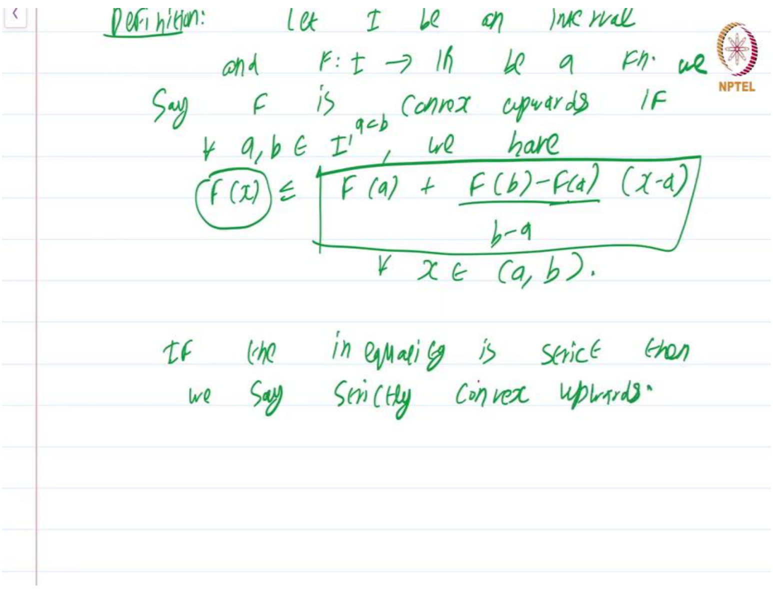

Let us define this formally definition. Let I be an interval, I am not specifying whether it is closed, open, half open, half close; nothing like that and $f: I \longrightarrow R$ be a function. Let $a, b \in I$ or rather we will say; we say f is convex upwards if for all $a, b \in I$, we have $f(x) \leq f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$; this is for all $x \in (a, b)$ ok.

So, its convex upwards if the value of the function in any sub interval is less than or equal to the line; the line joining the points $(a, f(a))$ and $(b, f(b))$, whose equation is given by this. So, this must be true for any pair of points $a, b \in I$. So, to clarify, I can just write $a < b$ ok; choose $a, b \in I$, such that $a < b$ this must be satisfied. If the inequality is strict, then we say strictly convex upwards.

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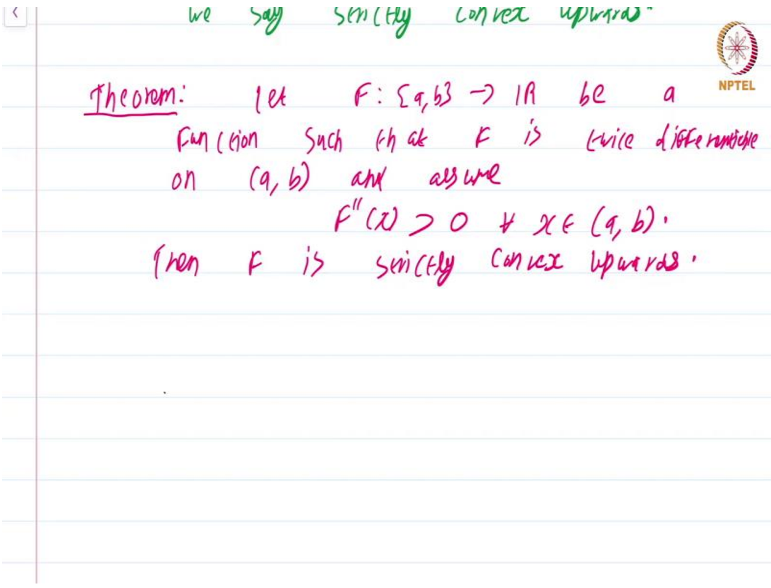

So, now let us prove a theorem that relates this notion of convex upwards with the second derivative. So, just let us see the picture again; if the function is sort of convex upwards, then observe that if you look at the various tangent lines to the curve; those seem to be sort of moving towards the left right.

So, this will be captured by the second derivative being greater than 0. So, let us see that. Theorem: let $f: [a, b] \longrightarrow R$ or I do not actually require, let it be let $f: [a, b] \longrightarrow R$ be a function such that f is twice differentiable on open interval (a, b) and assume $f''(x) > 0$ for all $x \in (a, b)$. Then, f is strictly convex upwards ok.

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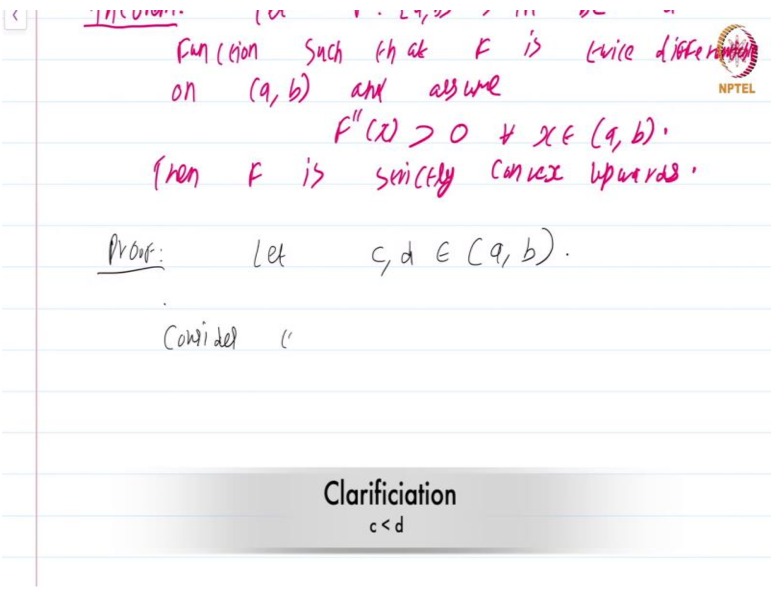

Now, proof let $c, d \in (a, b)$, ok.

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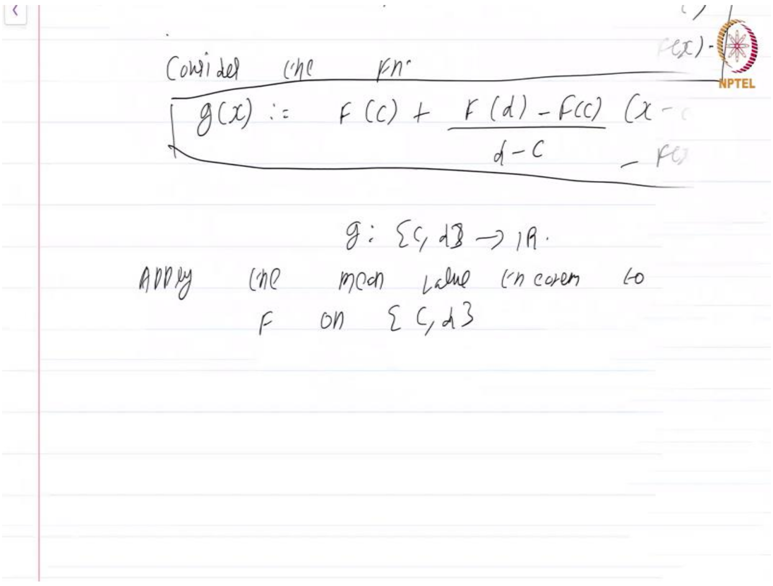

Now, consider the function $g(x) \coloneqq f(c) + \frac{f(d)-f(c)}{d-c}(x - b) - f(x)$. Now, this function should ring a bell; this is very similar to the function that we considered while proving the mean value theorem. So, what I am essentially doing is; I am considering the function, the line passing through the point $(c, f(c))$ and $(d, f(d))$ and I am subtracting $f(x)$ from this.

As can be guessed our goal is to show that this function $g(x)$ is greater than 0; for all points $x \in (a, b)$ ok. So, this defined this function $g: [c, d] \longrightarrow R$ ok. Now, how am I going to show that this function g is actually strictly greater than 0? Well, consider not consider; apply the mean value theorem to f on the closed interval $[c, d]$.

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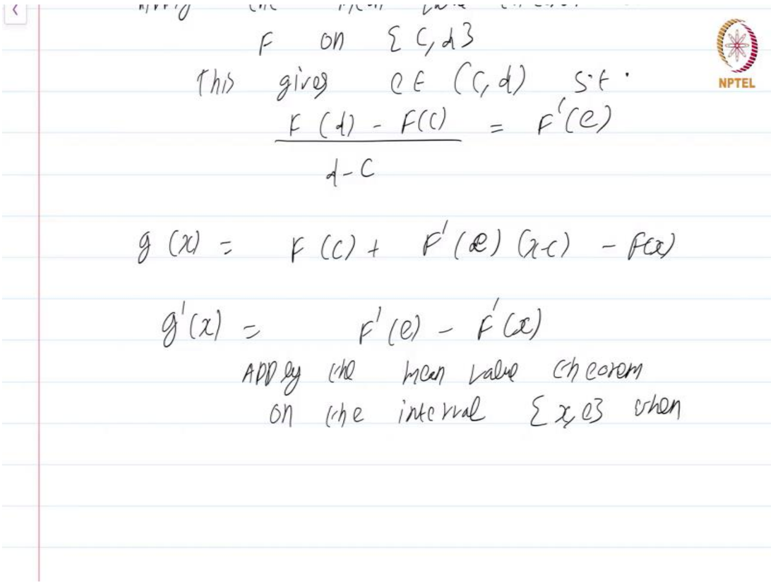

This gives $e \in (c, d)$ such that $\frac{f(d)-f(c)}{d-c} = f'(e)$ ok. So, consequently $g(x) = f(c) + f'(e)(x - c) - f(x)$. Therefore, $g'(x) = f'(e) - f'(x)$ ok. Now, observe that e is in between the point c and d. So, apply the mean value theorem; the mean value theorem on the interval, on the interval $[x, e]$; when $x < e$.

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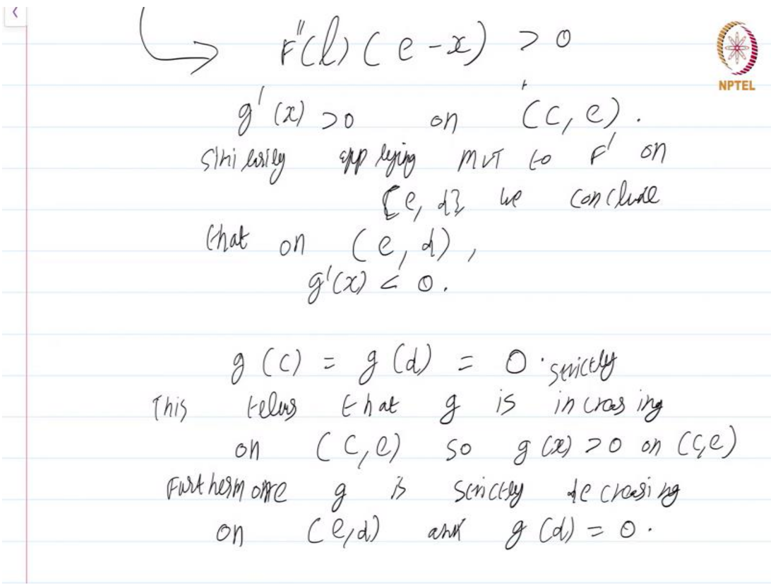

Again, so we are applying the mean value theorem again which we can do because we know that the second derivative exists ok. So, because of this you will get $f'(e) - f'(x)$ is some $f''(l)(e - x)$, ok. Now, this has got to be greater than 0 because $e - x > 0$ and f'' by hypothesis is greater than 0. My l looks like an e; so, let me just change it ok. So, this is greater than 0.

So, what we have concluded is that $g'(x) > 0$ on the interval $[c, e]$; x could have been any point in the interval; $[c, e]$ and we could have applied this argument to conclude the $g'(x) > 0$ here ok.

Similarly, applying mean value theorem to f' on the interval; $[e, d]$, we conclude that on $[e, d]$; $g'(x) < 0$ ok. So, what we have done is; we have applied the mean value theorem for a point x that lies between c and e to the function f' and concluded that $g'(x)$ has to be greater than 0 on $[c, e]$.

Similarly, we apply the argument to the point e; any point x coming from the open interval $[e, d]$ and we get $g'(x) < 0$. Now, how does this help us? Let us go back and refresh our memories as to what this function g is.

Observe that at the point $x = c$; you get a grand 0 right, you get a grand 0 and observe again that at the point $x = d$, you get a grand 0 yet again. Please check that, when $x = d$ or $x = c$; you get 0. So, $g(c) = g(d) = 0$. Now, what does this tell us?

This tells us that g is increasing on the interval $[c, e]$. So, in fact, strictly increasing because we have concluded that $g' > 0$; so it is strictly increasing on $[c, e]$. So, $g(x) > 0$ is greater than 0 on c right; so, by continuity and for furthermore, g is strictly decreasing on the interval $[e, d]$ and $g(d) = 0$ ok; that we already know.

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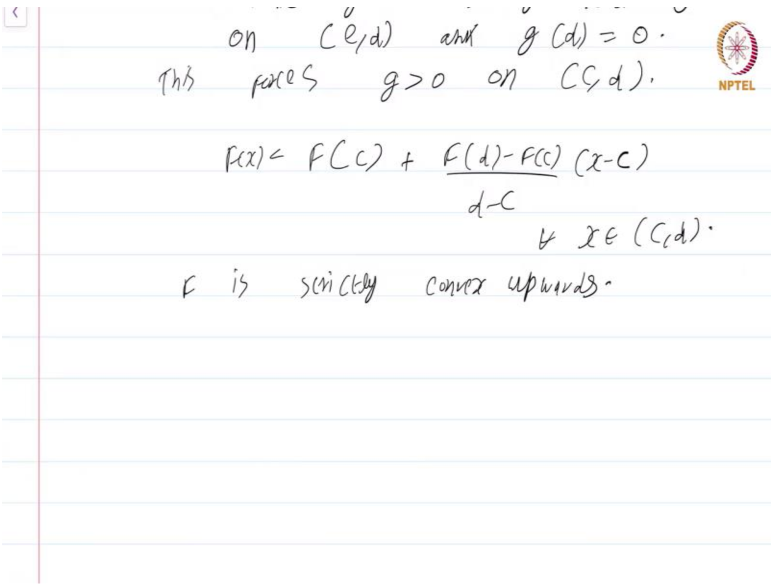

This forces, $g > 0$ on $[c, d]$. Why is that? $g(c) = 0$; the function is increasing from c to e, so when it reaches c; it will obviously, be strictly greater than 0 again. And from there, it can never reach 0 in the interval $[e, d]$ simply because its decreasing and at the end point d, it is still 0.

So, this forces $g > 0$ on $[c, d]$ including at the point e, you might think that the point e is an exception, but the argument we have given rules out the possibility that $g(e) = 0$ as well.

So, $g > 0$ on $[c, d]$. In other words, the function we considered $g(x) > 0$ which is just going to say that $f(x) < f(c) + \frac{f(d)-f(c)}{d-c}(x - c)$ b for all $x \in [c, d]$.

In other words, f is strictly convex upwards. So, this proof required a little bit of delicate analysis; we had to apply the mean value theorem twice, but now we know that the derivative, second derivative; the sign of the second derivative can help us determine how the function is sort of turning ok.

So, I leave it to you in the exercises to formulate versions of this when you have other sorts of inequalities, instead of greater than; you have less than f double prime is less than 0, less than or equal to 0, greater than or equal to 0; these are all trivial modifications of this. So, this concludes one set of applications of the mean value theorem. We will see some more later on; especially Taylor's theorem.

This is a course on Real Analysis, and you have just watched the module on Applications of the Mean Value Theorem.