Real Analysis - I
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Lecture – 23.2
The Mean Value Theorem

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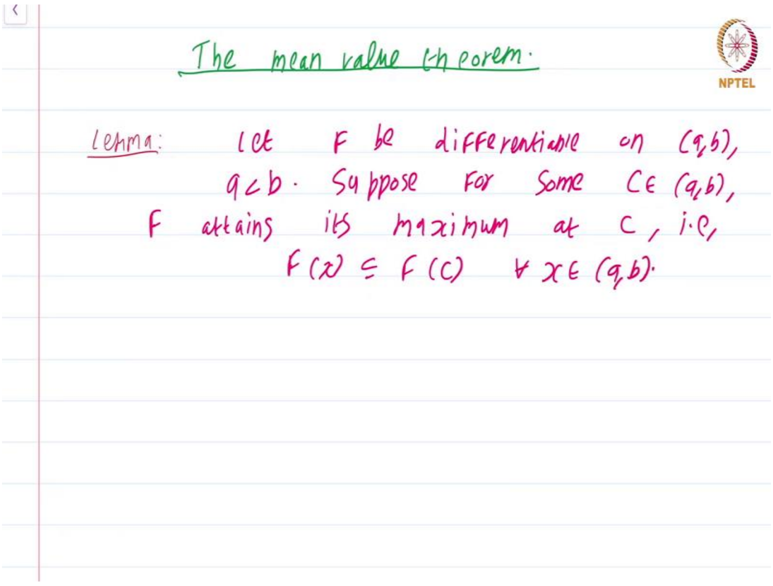

Almost all the properties of differentiation is a consequence of the Mean Value Theorem. So, we will first prove the mean value theorem and then, in a dedicated module, prove the various consequences. Then, in another module we shall see the most powerful consequence of the mean value theorem Taylor's theorem. So, let us begin with a simple lemma. Let f be differentiable on open interval(a, b) ; $a < b$ just to ensure that it is not the empty set.

Suppose, for some $c \in (a, b)$, f attains its maximum at c. What I mean by this is, that is $f(x) \leq f(c)$ for all $x \in (a, b)$; this is a point of maximum of the function f. Then, $f'(c) = 0$

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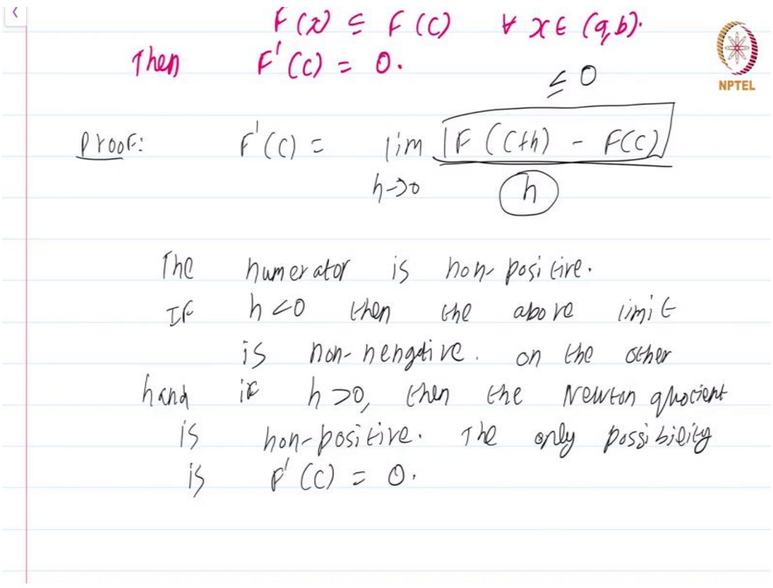

The derivative of a function at a point, where it attains its maximum in the open interval(a, b) is always going to be 0 and the proof of this is directly follows from the very definition of the Newton quotient. So, let us take the derivative at c. So, this $f'(c) = \lim_{h \to 0} \frac{|f(c+h)-f(c)|}{h}$.

Now, note that the numerator is always going to be less than or equal to 0. The numerator has to be negative. Why is that? Because the point of maximum is at c; so, the numerator is non-positive that is the correct way to say it; the numerator is non-positive. But the denominator could be positive or negative depending on whether h is greater than 0 or h is less than 0.

So, if $h < 0$, then the above limit is non-negative; because the numerator is non-positive and the denominator is negative, the quotient will have to be non-negative. On the other hand, if $h > 0$, then the quotient, the Newton quotient is non-positive. Because the denominator is positive; whereas, the numerator is non-positive therefore, the quotient will have to be non-positive.

The only possibility is $f'(c) = 0$. As you approach from the right and the left, the signs are different. Therefore, at the point c, the only possibility is that $f'(c) = 0$. ok. That was rather easy we immediately get a nice consequence of this which you might have heard is called Rolle’s theorem and says the following.

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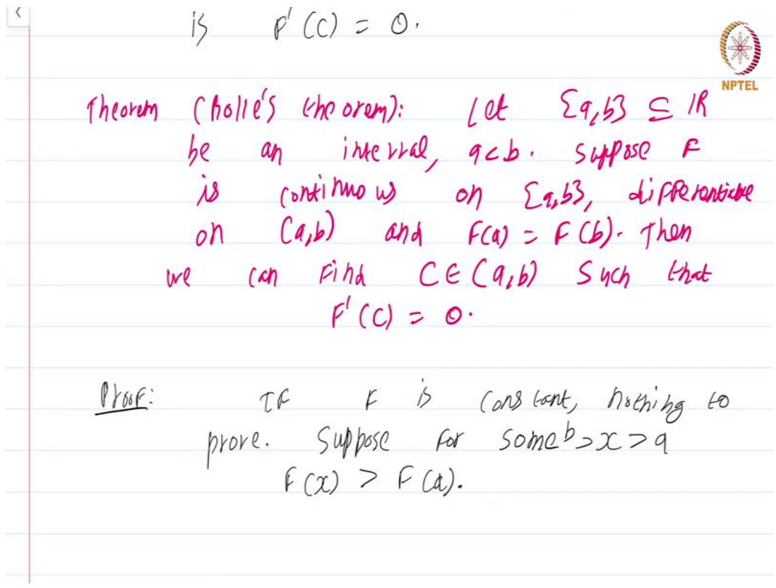

Let $[a, b] \subset R$ be an interval $a < b$. Suppose, f is continuous on the close interval$[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$. So, you have a function which is continuous on the closed interval $[a, b]$; differentiable on the open interval (a, b) and such that $f(a) = f(b)$; the values at the endpoints coincide. Then, we can find we can find $c \in (a, b)$ such that such that $f'(c) = 0$.

Proof; again, the proof is going to be a straightforward application of the previous lemma. If f is constant, nothing to prove, ok. Now, suppose for some $b > x > a, f(x) > f(a)$. We are assuming now that the function is non-constant. first, we make the assumption that at some point in the open interval (a, b); $f(x) > f(a)$.

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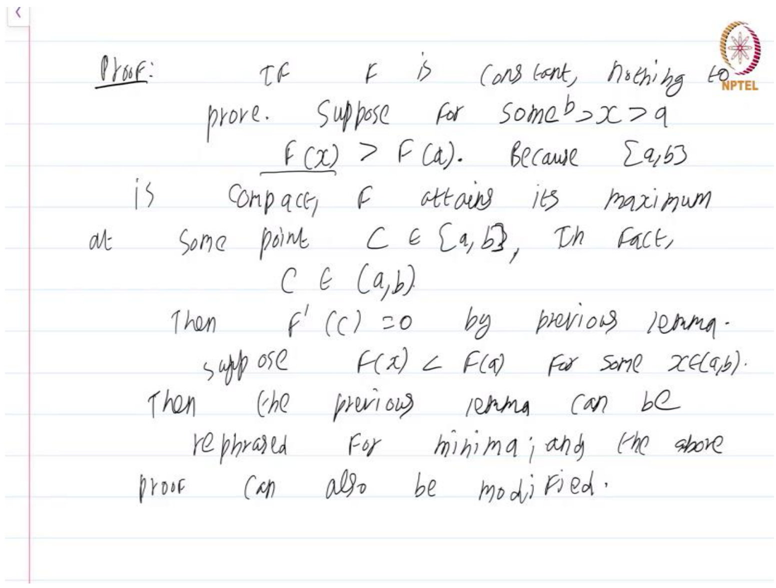

Because $[a, b]$ is compact, f attains its maximum somewhere at some point at some point $c \in [a, b]$; but the end points the values are equal and we already know that $f(x) > f(a)$. So, in fact, c must belong to the open interval (a, b) because of this the end points the values are same somewhere in the interior at the point x, its greater than $f(a)$; there must be a point $c \in (a, b)$ on which the maximum is assumed.

Here, I am using the fact that continuous functions on compact sets attain its maxima and minima right. I am just going to use the maximum. Then, $f'(c) = 0$ by previous lemma, ok. Now, the other case is, suppose $f(x) < f(a)$ for some $x \in (a, b)$; then, the previous lemma can be rephrased for minima.

The same proof with a very minus modification will tell you that at a point of minima also the derivative is 0, exact same proof just slight modification for minima and the above proof can also be modified.

So, this will be a pattern in even in this module as well as the next few ones, where I will just phrase it for one case. All the other cases are really straightforward and at this stage, you should be mature enough to be able to prove them on your own without even opening your eyes ok. So, this is done. Hence, proved; hence, proved.

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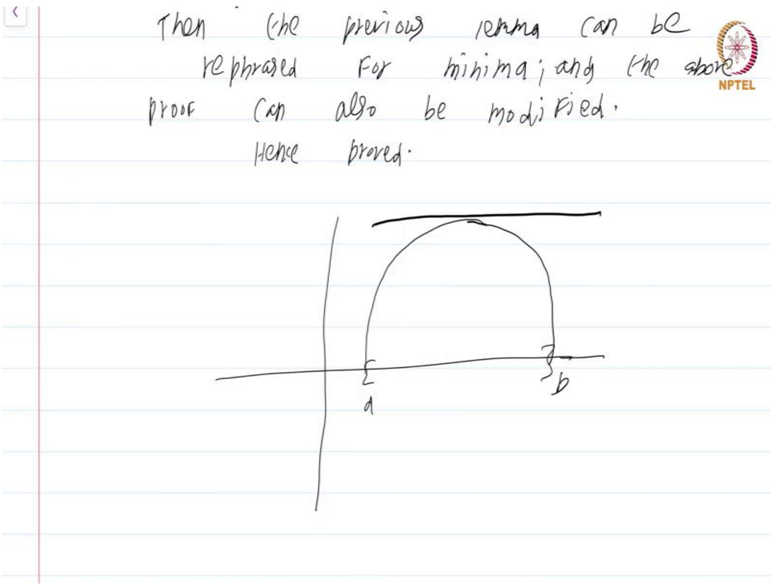

So, this is a geometrically obvious theorem again that requires a proof. So, this is what Rolle's theorem is saying if you have the point (a, b) and you have that the graph of the function returns to where it is. The way I have drawn it $f(a) = f(b) = 0$; but that does not really matter. All this is saying is if the values have to return, then the curve has to at some point become flat and turn, at that point when you consider the tangent line and the slope of that tangent that has to be 0.

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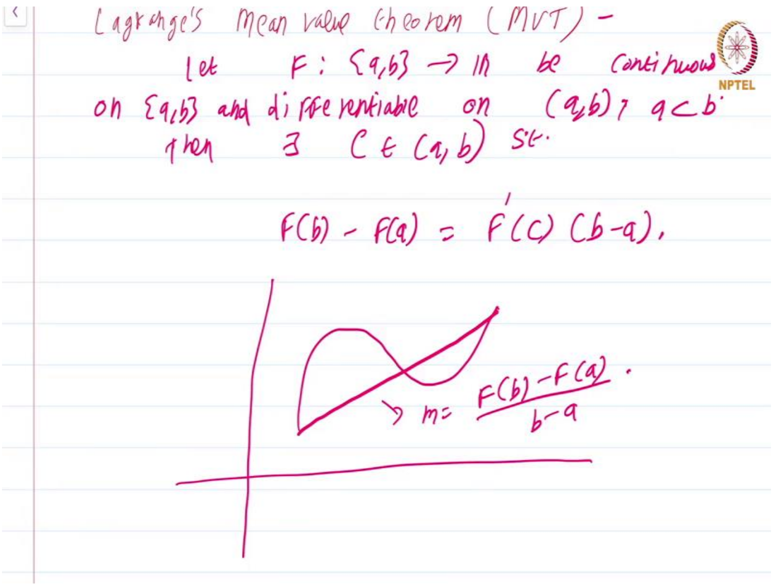

So, it is proving something that is geometrically obvious in a rigorous way. Now, we come to Lagrange's mean value theorem mean value theorem. Since this is going to be re again and again used, I will just abbreviate it as MVT. So that, there is economy. Let again $f: [a, b] \longrightarrow R$ be continuous and differentiable. Since this is an important theorem, I am not going to leave out any hypothesis. Let $f: [a, b] \longrightarrow R$ be continuous on $[a, b]$ and differentiable on open interval (a, b) ok. Again, $a < b$; then, there exists a point c in the open interval (a, b) such that $f(b) - f(a) = f'(c)(b - a)$ ok. Now, this result might seem mystifying and you can immediately dispel this mysticism by a simple picture. So, suppose you have some function; suppose you have some function like this. Now, this result if you have $f(b) = f(a)$, the left hand side is 0.

Therefore, this result is actually proved as Rolle's theorem when $f(b) = f(a)$; but we might not have $f(b) = f(a)$ right in general. So, what this is saying is if you join the slope of this line, the slope is $\frac{f(b)-f(a)}{b-a}$. This is from high school analytic geometry; the slope of the line is given by $\frac{f(b)-f(a)}{b-a}$.

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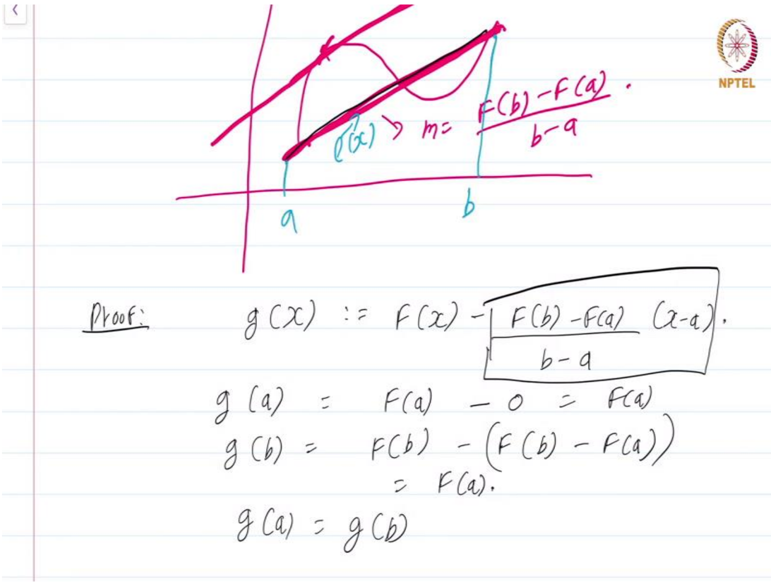

Now, what we are going to do is, we are going to view this function, sort of as a graph as a graph over this line. We are going to view this function as a graph over this line and then, apply Rolle's theorem ok.

So, all this is saying is if I view this function as a graph over this line, there will be a point where the slope will be parallel ok. So, if I am viewing this as a graph; that means, that functions derivative at this point it is actually 0 fine. So, this is a bit vague; but the proof will make it very clear what is happening proof.

Now, we want to apply Rolle's theorem to a new function. So, we define$g(x) \coloneqq f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. So, we have defined the new function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$.

Then, $g(a) = f(a) – 0 = f(a)$ and $g(b) = f(b) – \big(f(b) - f(a)\big) = f(a)$.

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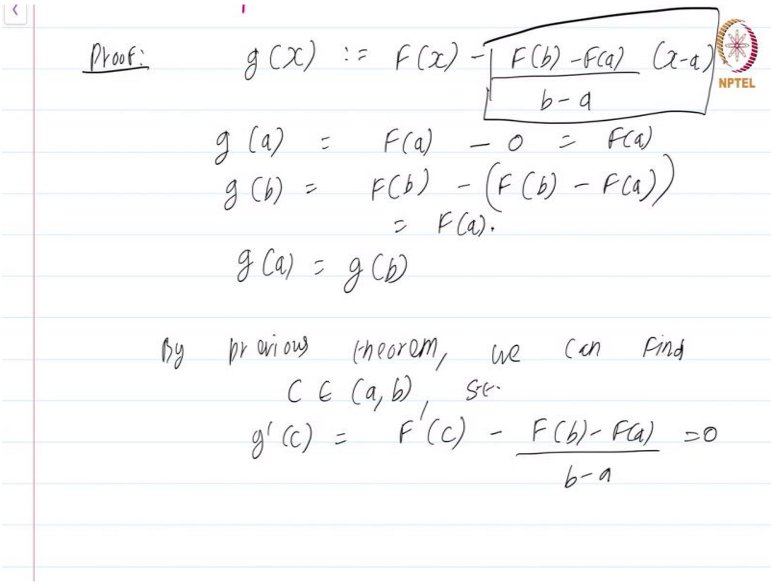

So, what we have managed to do is we have managed to make $g(a) = g(b)$, by subtracting this $\frac{f(b)-f(a)}{b-a}(x-a)$ right. So, actually, to illustrate the point that I am going to make this a graph over this particular line, it is better to have subtracted a slightly different function; but this will do the job for us, this will do the job for us ok.

Now, by the previous lemma or previous theorem that is Rolle's theorem, by previous theorem, we can find c in the open interval (a,b) such that $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$ or in other words, $f(b) - f(a) = f'(c)(b-a)$.

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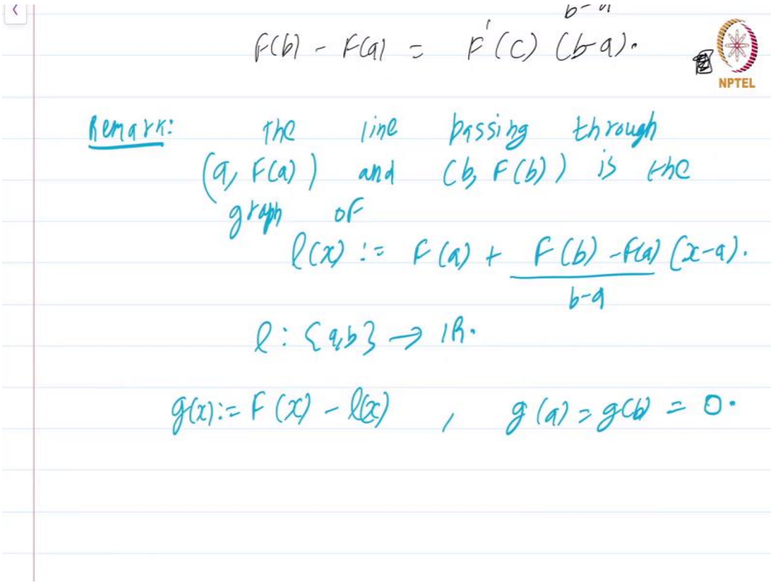

So, let me clarify one remark I made earlier that this function will do the job for us, but a better function is to consider, I mean we are done with the proof; I am just going to make some extra remarks.

So, let me just put a box. So, remark, the line the line passing through $(a, f(a))$, and $(b, f(b))$ is the graph of is the graph of $l(x) := f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ ok. This is actually going to be the graph; $l: [a, b] \longrightarrow R$. This $l(x)$ is actually the equation of this line, of this line from a to b ok.

So, alternatively we could have this is almost exactly same as the function we subtracted, except one slight change will happen. If you consider $f(x) - l(x)$, you would have landed up with the symmetric situation and call this the function g. You would have ended up with $g(a) = g(b) = 0$, ok.

You would have landed up with this nice symmetric situation, where $g(a) = g(b) = 0$, that is the only difference. fine. You could have applied the Rolle's theorem now and concluded the proof. So, from the next module onwards, we shall see applications of the mean value theorem.

This is a course on Real Analysis, and you have just watched the module on the Mean Value Theorem.