Real Analysis - I
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Lecture - 2.4
Proofs

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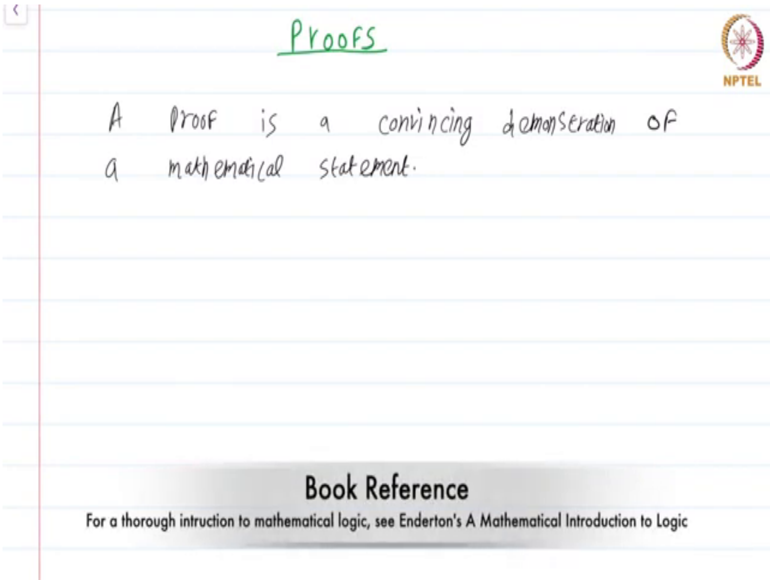

We now come to the heart of these modules, logic - Proofs.

Ultimately the entire development of logic was to make precise what proofs are. So, proof is a convincing demonstration of a mathematical statement. Now in the previous modules, we have seen what a statement is. It is a statement that has some sentences which have true or false values. These sentences could depend on a variable, they could have quantifiers attached to them. We have to somehow determine the truths.

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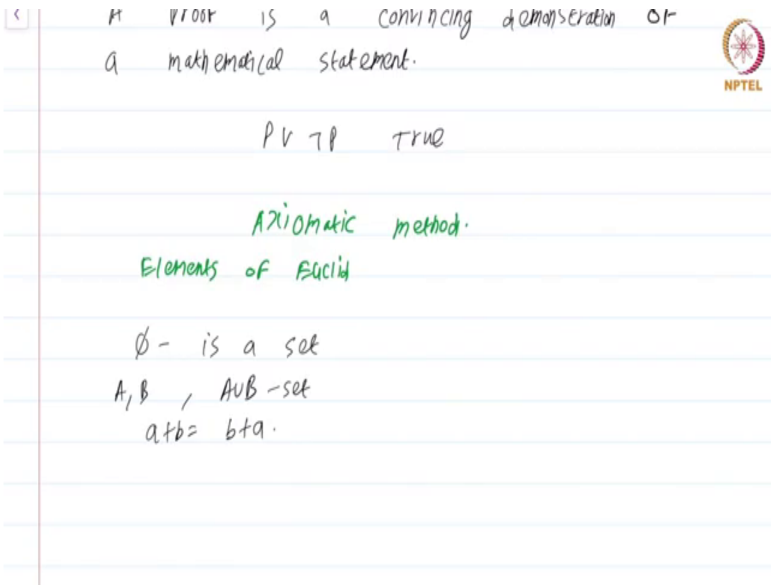

Now, one set of statements that are; obviously, going to be true are tautologies which you have already studied. There are several tautologies. For instance no matter what type of statement you take P or negation P is always going to be true. This is always going to be true, just by the very definitions of the meaning of the letter or of the connective or.

Now, in mathematics things are rarely so simple. We have to study things like real numbers, then more complicated things like differentiation, integration etc. and prove theorems about these, how one goes about that is known as the axiomatic method. The axiomatic method, it is unknown who exactly pioneered the idea, but the earliest work that is surviving till today and has an incredible exposition, incredible demonstration of this method is the elements of Euclid. It is inspired by elements of Euclid.

Elements of Euclid is a text written somewhere in the 3rd century BC; that has a systematic development of classical geometry starting from various axioms. An axiom is supposed to be a self-evident truth. Later in the 19th and 20th century, it turned out that axioms are not necessarily self evident truths. They are just statements that we take as true for some reason, when you say something is self-evident; that means, it is incontestable.

One of the axioms of Euclid was that parallel lines do not meet. Is that self evident? I do n't think so. We can never really check whether that is true in the real world. In fact, modern theories of physics suggest that in the real universe, this might not be satisfied.

So, that is just an assumption that Euclid has made explicitly and developed geometry from that, it is still useful in the real world because for all practical purposes, it looks like parallel lines do not meet. If at all they do meet, they meet at a very very far away distance. It is really immaterial in day to day considerations.

So, there are alternative geometries developed in the 19th and 20th centuries where this assumption that parallel lines do not meet is no longer made. So, for the purposes of this course, axioms are merely explicit statements that we take for granted . One of the axioms of set theory that we will see in a future module is that the empty set is a set phi; the empty set is a set.

Another axiom that we will see is that if A and B are sets, A union B is a set. And another yet another axiom, we will see when we study the real numbers is that in the real numbers a plus b is equal to b plus a . So, the idea behind Euclid's elements or more generally, the axiomatic method is you begin by setting once and for all what your assumptions are and you do not change them in the future. You fix what your assumptions are and you try to derive everything from these assumptions.

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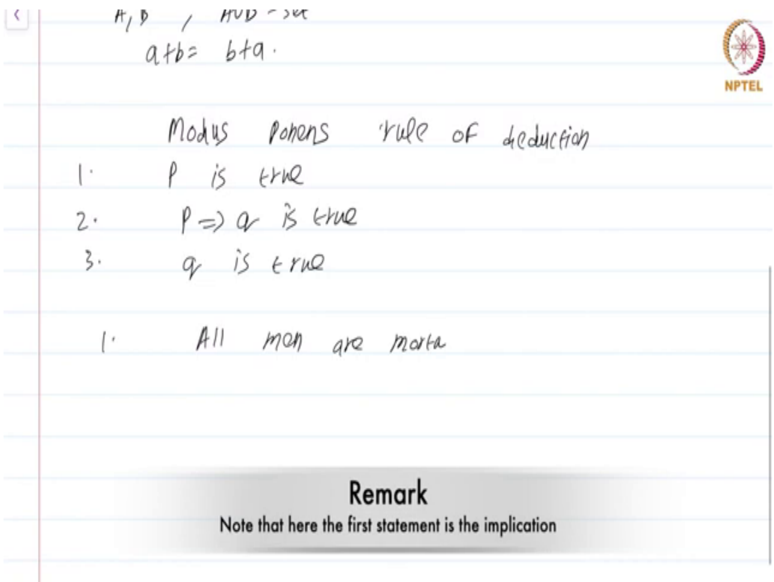

So, what is a derivation? Well, since this is not a course on logic. I will be very brief the classical derivation is using what is known as modus ponens is what is known as modus ponens. What it says is something really simple. It says the following. Suppose you have demonstrated that p is true, a statement particular statement p is true. You have demonstrated it in some way, it could also be an axiom; axioms are taken to be true.

Second, suppose you have demonstrated that p implies q is true . What modus ponens says is that then q is true this is called a rule of deduction. This is a rule of deduction, if you have demonstrated that p is true and p implies q is true, then q is true. It is famously illustrated in the following way all men are mortal. This is taken to be true.

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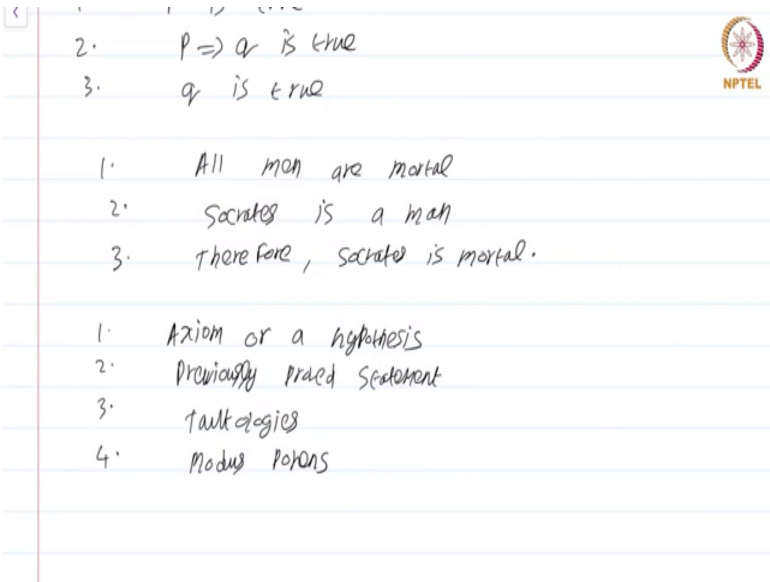

Second, Socrates is a man. Three, therefore Socrates is mortal. This is the classical illustration of this rule of deduction called modus ponens. Now, this rule is so ingrained in mathematicians that it is highly unlikely that any textbook when they are writing a proof, they will mention that modus ponens is being applied.

It is taken for granted, it is such an ingrained and straightforward rule that it is taken for granted. You never have to mention when you are applying modus ponens, but surprisingly most of mathematics is just application of this single rule of deduction and variants of these.

So, what does a proof do? Well, it is a sequence of statements; it is a sequence of statements. Each statement is either an axiom or a hypothesis whenever you state a theorem. If the statement goes that if such and such a thing is true, then we have the following conclusion. So, each step is either an axiom or a hypothesis or a previously proved statement that is also allowed. You are also allowed to invoke tautologies anywhere you want tautologies.

And then finally, you can also use modus ponens. Each step should either be an axiom or a hypothesis. It can be a previously proved statement. It can be a tautology or it can be a deduction done through modus ponens . In reality when mathematicians write, we do not want to overwhelm the reader by giving an excessively wrong proof that is in the sequential order.

Usually we will write the proof in English and several steps will be combined into one. It is not that beneficial to write out the proofs in entire detail, but rather just you develop the practice just by seeing a statement that is present in a proof to see how it was derived. You will pick up this skill by practice and when I mean practice, I mean not seeing theorems, but actually sitting down and proving theorems .

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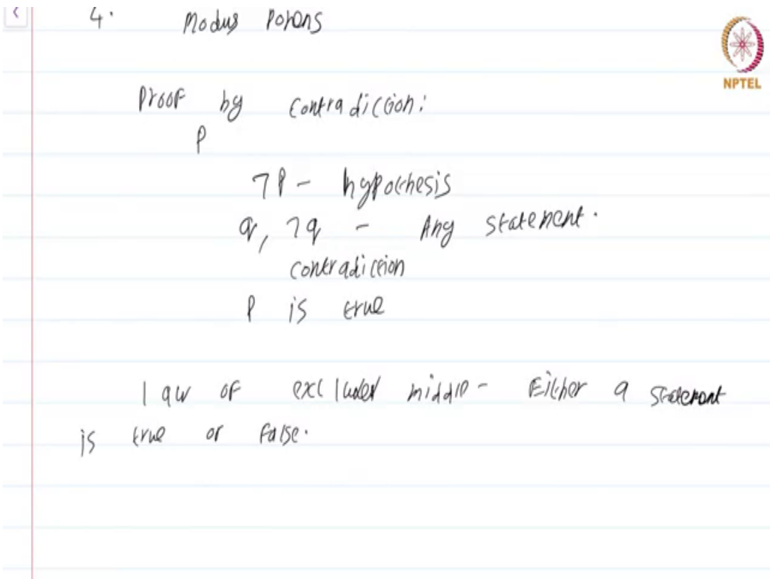

So, just one more remark, I want to make it about proof by contradiction. This is a very very powerful technique that is used extensively in classical as well as modern mathematics. Suppose you want to prove a particular statement p to prove p instead of directly proving p, what you do is you consider the statement negation of p and considered that also as a hypothesis. In essence you are assuming that negation p is true when you write down a proof, what you do is in each of those steps where you are allowed to use a hypothesis, you can use negation p also in its place.

And your aim is to prove both statements q and statement negation q, where q could be any statement. So, in the proof by contradiction, the idea is to start by assuming negation of p as a hypothesis and deriving proofs of q as well as negation q. This is what is known as a contradiction.

A contradiction is a proof that manages to prove both the statement q as well as the negation q, contradictions are not allowed. Therefore, negation p cannot be true therefore, p must be true. So, this will allow us to conclude that p is true instead of directly proving p is true, you arrive at a contradiction by assuming negation of p and that gives you the proof that p is true.

Why does this work? It works because of the law of excluded middle. The law of excluded middle is just a fancy way of saying either a statement is true or false; either a statement is true or false.

Note, this was just our definition of statement. A statement is a sentence that has either a value true or a value false, you can assign a value to it. So, a statement cannot be both true and false. It cannot be neither true nor false. Those are things that we do not study in classical logic. They are; however, studied in many forms of logic including traditional logic in India, but it is not the content of logic that we use.

So, what this proof by contradiction does is, we manage to prove both statement q as well as negation q that is simply not possible, both cannot be true. Therefore, it must be the case that the original hypothesis that negation p is true ,cannot be true. Therefore, p must be true. Assuming that p is true, we arrive at statement q which is both true as well as the negation is true which is simply impossible by the definition of a statement. Therefore, p must be true; this is the idea behind proof by contradiction.

There is a somewhat related idea called 'Proof by Contrapositive', but that is a direct proof. Let me tell you what proof by contrapositive is; in most of mathematics as I have mentioned repeatedly. You have some hypothesis and you want to prove a conclusion.

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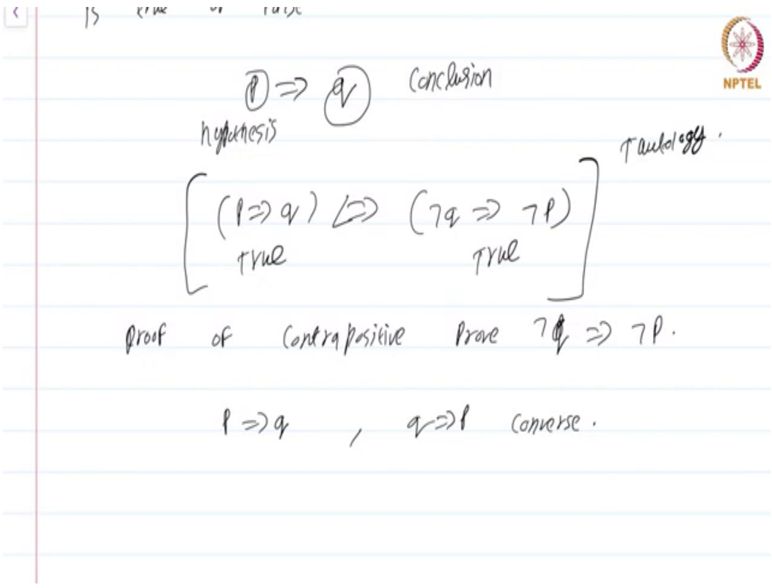

So, usually the statements that you have to prove in mathematics are of the type p implies q, where p is supposed to be your hypothesis and q is the conclusion you want. So, usually what you are trying to prove is under a certain hypothesis, the statement q is true that in other words, you are trying to prove that p implies q.

Now, note that or rather check that p implies q is equivalent to negation q implies negation p. What do I mean by equivalent to? I mean that both truth tables are exactly the same. The truth table of p implies q and the truth table of negation q implies negation p are exactly the same. So, if you could show that negation q implies negation p is true then so, is this statement.

So, in the proof by contrapositive; what you do is instead of trying to prove that p implies q what you do is you prove negation q implies negation p. So, the proof by contrapositive just uses the fact that this entire statement is a tautology, that this is a tautology .

If you can show that p implies q if and only if negation q implies negation p and you show negation q implies negation p. Then by modus ponens, you get p implies q that is the derivation proof of this alternatively just write the truth table and check that p implies q and negation q implies negation p have exactly the same truth tables. So, proof by contrapositive many students do confuse it with proof by contradiction, because there is negation involved; proof by contrapositive is a direct proof. You do not need to assume this law of excluded middle to get proof by contrapositive, but those gets into subtleties. I do not want to deal with them right now .

Now, one more thing is the statement if you have shown p implies q, the statement q implies p is called the converse. This is called the converse. Now one mistake that students often make is to try to prove p implies q, they instead end up proving q implies p that is not always true. It is not true that p implies q and q implies p are equivalent statements; to check that write down the truth tables of p implies q and q implies p and see that they are not the same.

This concludes the series of modules on logic. This is a course on real analysis and you have just watched the module on proofs.