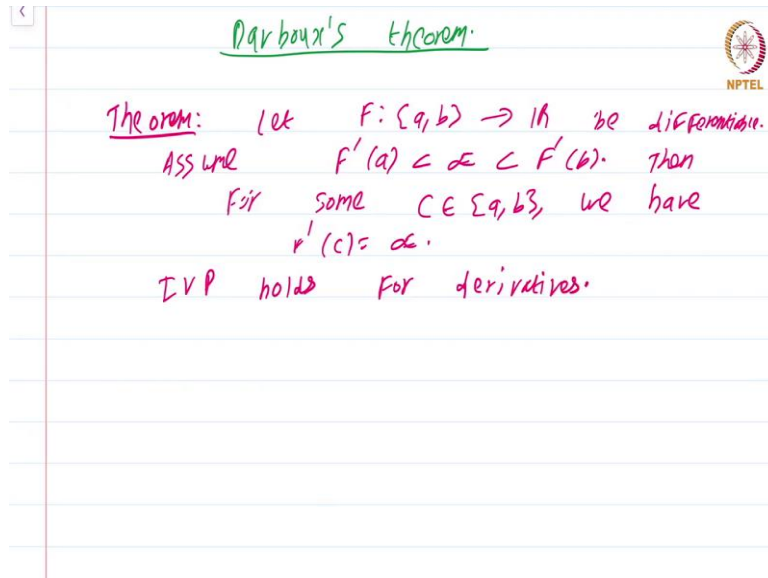


**Real Analysis - I**  
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**Lecture – 23.1**  
**Darboux's Theorem**

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The slide shows handwritten text on a lined background. At the top, 'Darboux's theorem.' is written in green and underlined. To the right is the NPTEL logo. The main text, written in pink, states: 'Theorem: let  $f: [a, b] \rightarrow \mathbb{R}$  be differentiable. Assume  $f'(a) < \alpha < f'(b)$ . Then for some  $c \in (a, b)$ , we have  $f'(c) = \alpha$ . IVP holds for derivatives.'

Now, I am going to prove one of my favourite theorems on the derivative. The theorem is as follows. Theorem, let  $f: [a, b] \rightarrow \mathbb{R}$  closed interval be differentiable including the end points, of course, differentiable. Assume that  $f'(a) < \alpha < f'(b)$ . Then for some  $c$  in the closed interval  $a, b$ , we have  $f'(c) = \alpha$ ; in other words, intermediate value property holds for derivatives ok.

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$f'(c) = \alpha.$   
DVP holds for derivatives.  
**Corollary:**  $f'(x): [a,b] \rightarrow \mathbb{R}$  cannot have a jump discontinuity.  
**Proof:** Consider  $g(x) = f(x) - \alpha x.$   
 $g'$  exists and  $g'(a) < 0, g'(b) > 0$   
Enough to show that  $g'(c) = 0$  for some  $c \in [a,b].$

Before the proof, let me state a corollary. Corollary,  $f'(x): [a, b] \rightarrow \mathbb{R}$  cannot have a jump discontinuity. Now, let us prove the theorem the proof of the corollary is so obvious that I am not even going to justify giving a proof. Let us consider this for a moment. We want to show that all intermediate values are taken.

So, what I do is I simplify the situation. Consider,  $g(x) = f(x) - \alpha x$  ok. Now,  $g'$  exists and we clearly see that  $g'(a) < 0$ , and  $g'(b) > 0$ . This just follows because the derivative of  $\alpha x$  is just  $\alpha$ . So, enough to show that  $g'(c) = 0$  for some  $c \in [a, b]$ .

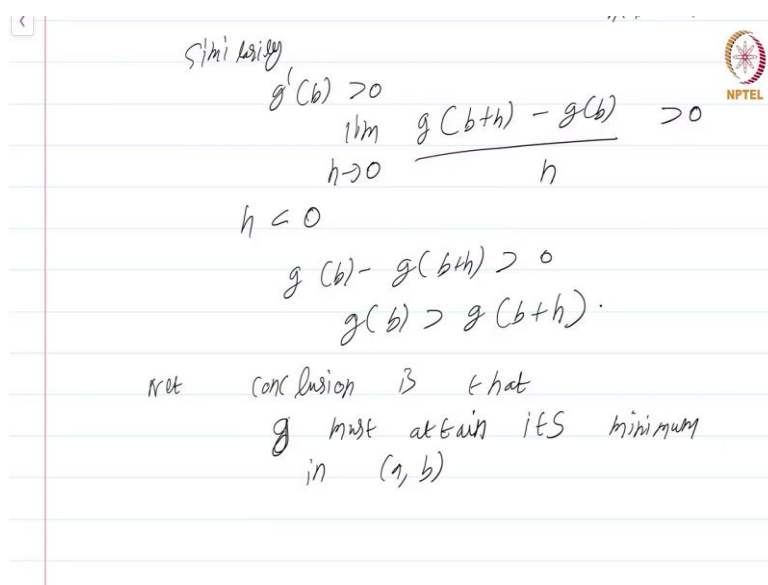
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$g'(c) = 0$  for some  $c \in [a,b]$   
since  $g'(a) < 0$   
 $\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} < 0$   
 $h > 0$   
For  $h$  close to 0 and  $> 0$   
 $g(a+h) < g(a).$   $g$  is "decreasing" near  $a.$

Let us look at what  $g'(a) < 0$  trying to say. Well since  $g'(a) < 0$ , that means,  $\lim_{h \rightarrow 0} \frac{g(a+h)-g(a)}{h} < 0$ . And note because  $a$  is the left endpoint of the interval,  $h > 0$  here. In other words, for  $h$  close to 0 and greater than 0, we get  $g(a+h) < g(a)$ , ok.

In other words,  $g$  is decreasing, I will put this in quotes near  $a$ . So, in a later module, once we prove the mean value theorem, we will show that this  $g$  is decreasing and the sign of the derivative being less than 0 are intimately related to each other.

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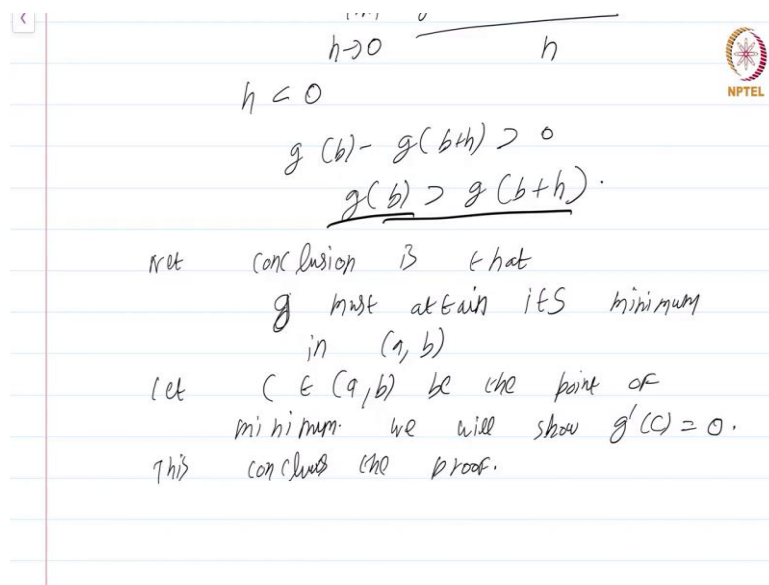
Similarly  
 $g'(b) > 0$   
 $\lim_{h \rightarrow 0} \frac{g(b+h) - g(b)}{h} > 0$   
 $h < 0$   
 $g(b) - g(b+h) > 0$   
 $g(b) > g(b+h)$   
 net conclusion is that  
 $g$  must attain its minimum  
 in  $(a, b)$

Similarly,  $g'(b) > 0$ , so that means,  $\lim_{h \rightarrow 0} \frac{g(b+h)-g(b)}{h} > 0$ , right, but here  $h$  has to be negative;  $h$  is negative simply because we are at the right end point ok. So, this just this will just show that  $g(b) - g(b+h) > 0$ , in other words  $g(b) > g(b+h)$ .

So, net conclusion is, net conclusion is that  $g$  must attain its minimum in  $[a, b]$ ;  $g$  certainly attains its maximum and minimum in the closed interval  $[a, b]$ ; it must attain its minimum in the open interval  $(a, b)$ . Why is that?

Well, because we have found that near the point  $a$  there is a point  $a+h$  such that  $g(a+h) < g(a)$ , and near the point  $b$  we have found a point such that  $g(b) > g(b+h)$ . So, both put together says that neither  $g(a)$  nor  $g(b)$  can be the minimum of the function  $g$ . So,  $g$  must attain its minimum in the in  $(a, b)$ .

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Handwritten notes on a slide with an NPTEL logo in the top right corner. The notes are written in black ink on a light blue background. The text is as follows:

$h \rightarrow 0$   $\frac{0}{h}$

$h < 0$

$g(b) - g(b+h) > 0$

$g(b) > g(b+h)$

net conclusion is that

$g$  must attain its minimum in  $(a, b)$

let  $c \in (a, b)$  be the point of minimum. we will show  $g'(c) = 0$ .

This concludes the proof.

So, let us  $c \in (a, b)$  be the point of minimum be the point of minimum ok. Now, in the next module, we will show; we will show  $g'(c) = 0$ , ok, that we will see in the next module. So, this will conclude the proof..

This is a course on real analysis. And you have just watched the module on Darboux Theorem.