Real Analysis - I

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Lecture – 23.1
Darboux s Theorem

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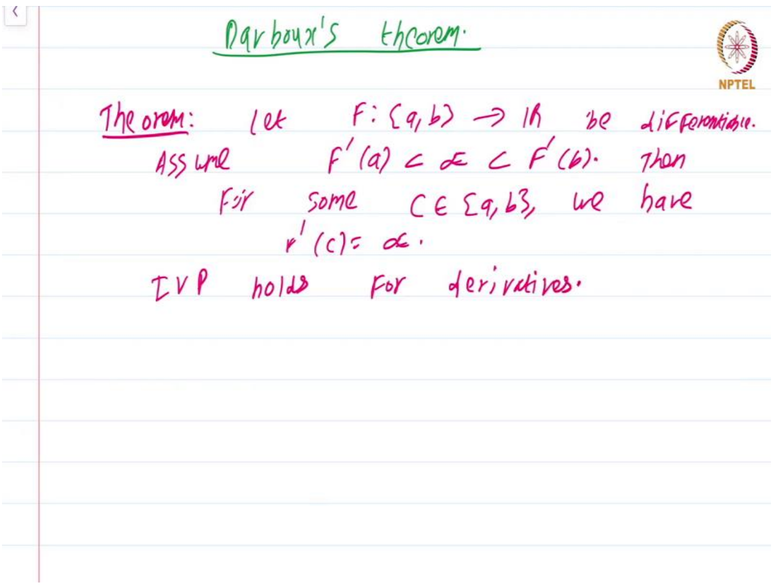

Now, I am going to prove one of my favourite theorems on the derivative. The theorem is as follows. Theorem, let $f: [a, b] \longrightarrow R$ closed interval be differentiable including the end points, of course, differentiable. Assume that $f'(a) < \alpha < f'(b)$. Then for some c in the closed interval a, b, we have $f'(c) = \alpha$; in other words, intermediate value property holds for derivatives ok.

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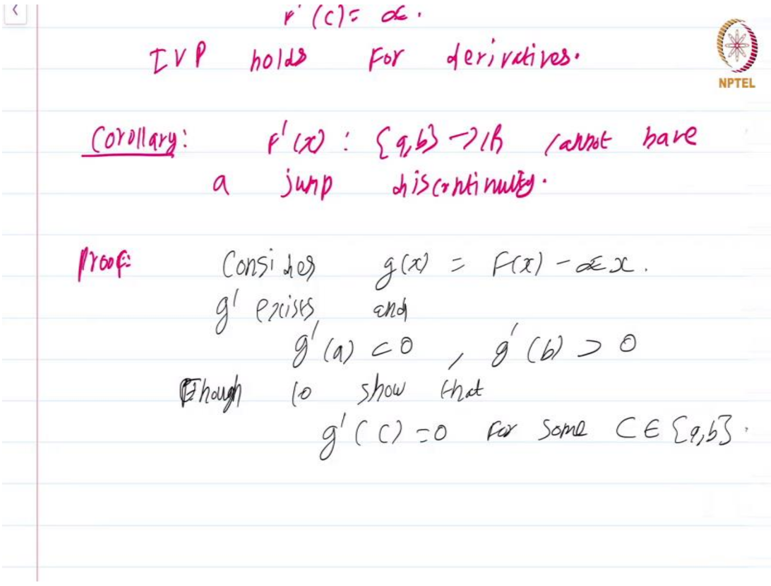

Before the proof, let me state a corollary. Corollary, $f'(x)\colon [a,b] \longrightarrow R$ cannot have a jump discontinuity. Now, let us prove the theorem the proof of the corollary is so obvious that I am not even going to justify giving a proof. Let us consider this for a moment. We want to show that all intermediate values are taken.

So, what I do is I simplify the situation. Consider, $g(x) = f(x) - \alpha x$ ok. Now, g' exists and we clearly see that $g'(a) < 0$, and $g'(b) > 0$. This just follows because the derivative of αx is just α. So, enough to show that $g'(c) = 0$ for some $c \in [a,b]$.

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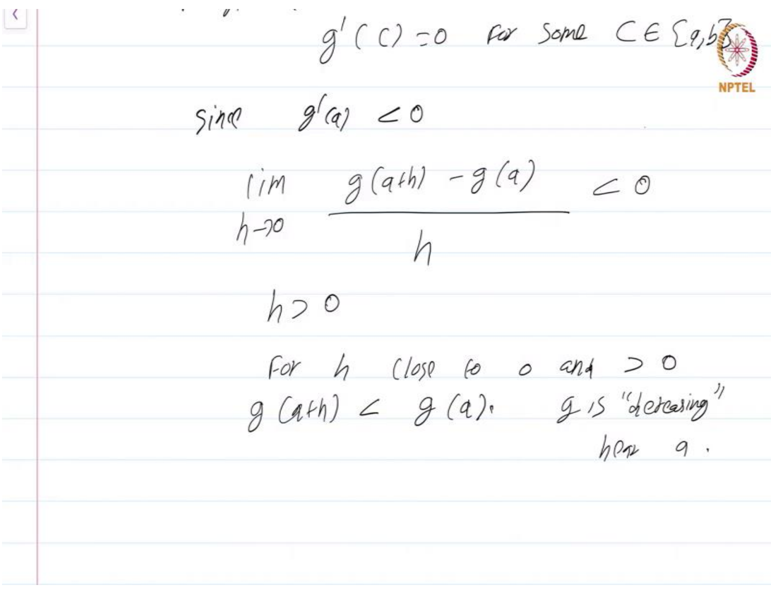

Let us look at what $g'(a) < 0$ trying to say. Well since $g'(a) < 0$, that means, $\lim_{h \to 0} \frac{g(a+h)-g(a)}{h} < 0$. And note because a is the left endpoint of the interval, $h > 0$ here. In other words, for h close to 0 and greater than 0, we get $g(a+h) < g(a)$, ok.

In other words, g is decreasing, I will put this in quotes near a. So, in a later module, once we prove the mean value theorem, we will show that this g is decreasing and the sign of the derivative being less than 0 are intimately related to each other.

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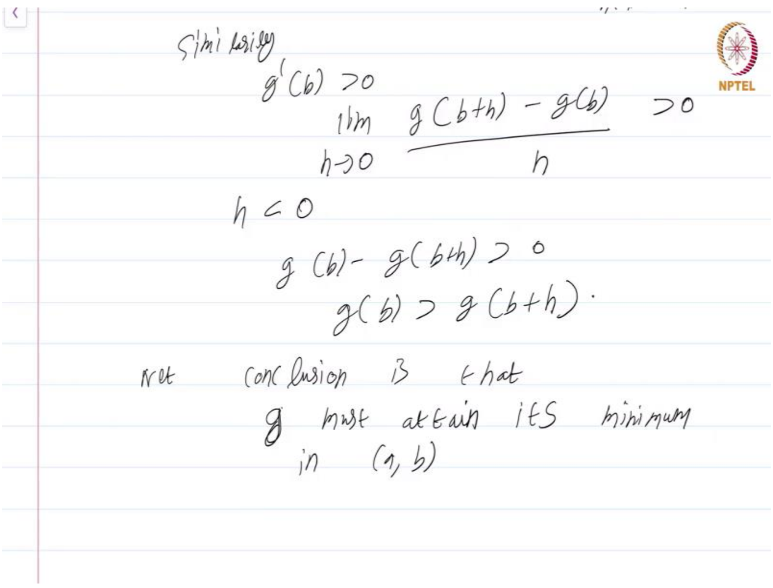

Similarly, $g'(b)$>0, so that means, $\lim_{h \to 0} \frac{g(b+h)-g(b)}{h}$ > 0, right, but here h has to be negative; h is negative simply because we are at the right end point ok. So, this just this will just show that $g(b) - g(b+h) > 0$, in other words $g(b) > g(b+h)$.

So, net conclusion is, net conclusion is that g must attain its minimum in $[a, b]$; g certainly attains its maximum and minimum in the closed interval $[a, b]$; it must attain its minimum in the open interval (a, b). Why is that?

Well, because we have found that near the point a there is a point $a + h$ such that $g(a+h) < g(a)$, and near the point b we have found a point such that $g(b) > g(b+h)$. So, both put together says that neither $g(a)$ nor $g(b)$ can be the minimum of the function g. So, g must attain its minimum in the in (a, b).

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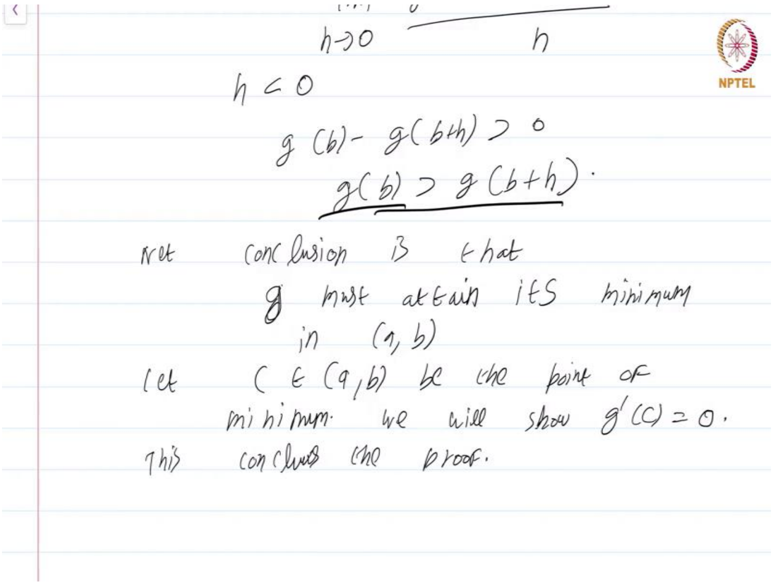

So, let us $c \in (a, b)$ be the point of minimum be the point of minimum ok. Now, in the next module, we will show; we will show $g'(c)$ = 0, ok, that we will see in the next module. So, this will conclude the proof..

This is a course on real analysis. And you have just watched the module on Darboux Theorem.