Real Analysis - I

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Lecture – 22.2
Basic Properties of the Derivative

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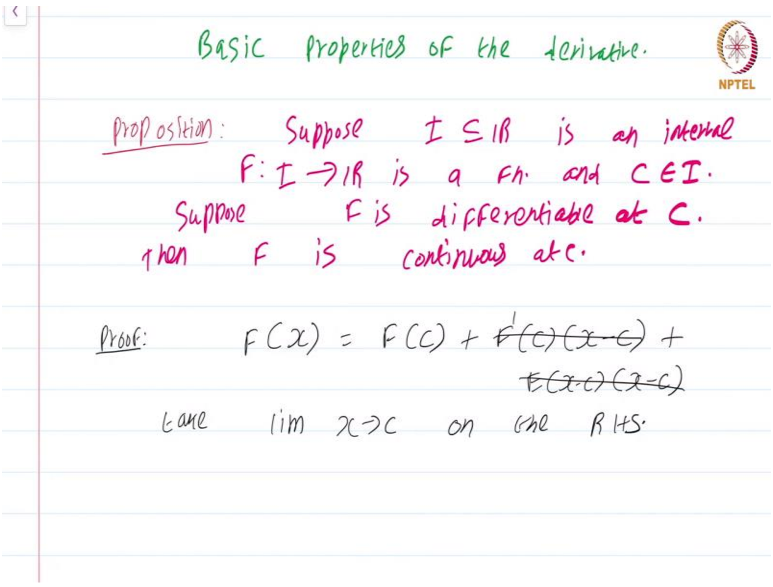

In this module, I shall prove some of the very Basic Properties of the Derivative. first, let us start with a very simple proposition; suppose $I \subset R$ is an interval; $f: I \to R$ is a function and $c \in I$. Suppose, f is differentiable at c. This just means as you can guess that the derivative at the point c exists. Then, f is continuous at c. The proof is fairly easy. Let us consider what the second interpretation of the derivative as a good linear approximation allows us to prove this quickly; let us see how it does that. We know that $f(x) = f(c) + f'(c)(x - c) + E(x - c)(x - c)$, ok. Now, take $\lim_{x \to c}$ on the RHS ok. This will vanish and so will this right.

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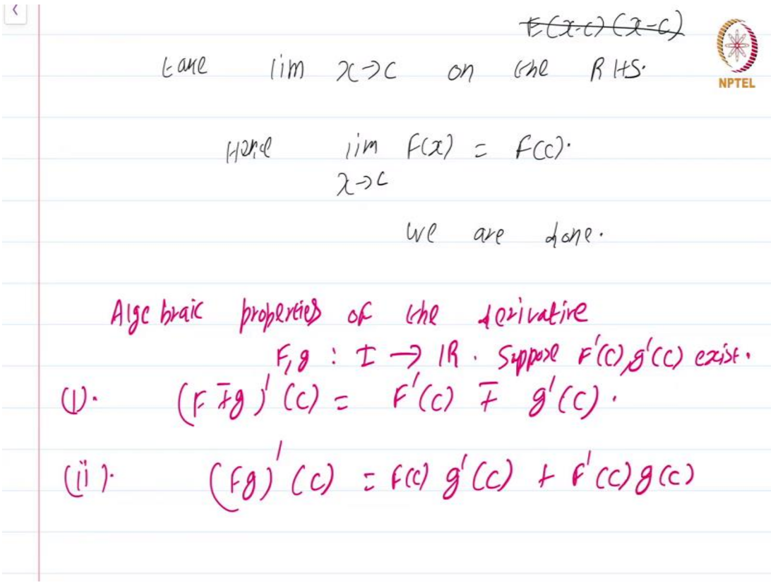

Hence, $\lim_{x \to c} f(x) = f(c)$; we are done. So, we got the proof instantly; once we use the interpretation of the derivative as giving a good linear approximation. Many other properties will also follow quite easily from this; let us consider the various algebraic properties of the derivative. Algebraic properties of the derivative.

This time f, g are functions from I to R, then suppose $f'(c), g'(c)$ exists. Then, $(f + g)'(c) = f'(c) + g'(c)$; in fact, I can put $\pm$ also. This one I am not even going to bother proving this is so utterly easy. Second, $(fg)'(c)$ $= f(c)g'(c) + f'(c)g(c)$, this is also called the Leibniz rule; let us prove this.

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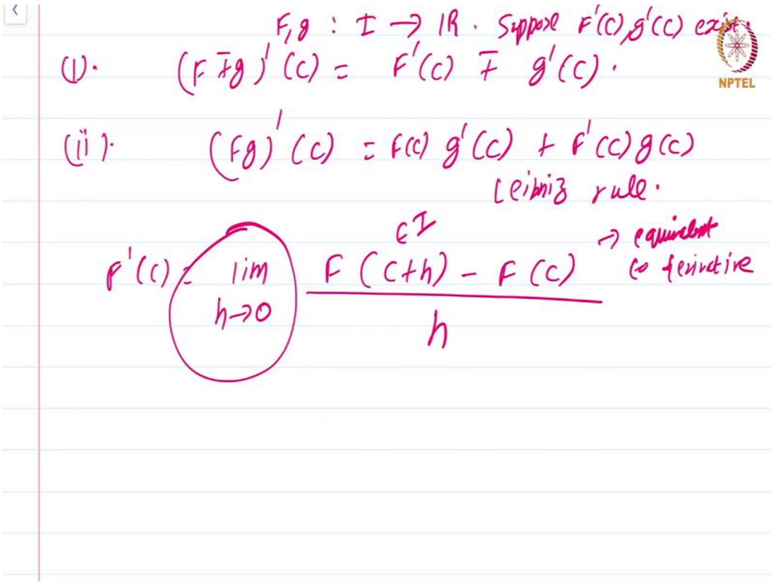

So, as is familiar; I will just rewrite $f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$. Now, one trivial remark, but still nevertheless has to be made is that the point c could be one of the end points of I.

I am not specifying whether I is an open interval or I is a closed interval, I am making no such distinction; I is just an interval and it can happen that the point c is one of the end points of the interval, in which case when I write limit h going to 0; it obviously means the corresponding right hand limit or left hand limit as the case might be.

As this is not such an important point, I will not be extra scrupulous and simply write limit h going to 0 with the understanding that the choices of h is to ensure that the $c + h$ is there in I ok. Always the h is chosen so that the $c + h$ is in I; this is implicit ok.

Now, I have just rewritten the usual definition of derivative in terms of the variable h; that this is equivalent to derivative is easy to see, equivalent to derivative is very easy to see. The usual definition involving $\frac{f(x)-f(c)}{x-c}$, as limit x going to c, this and this quotient will exactly be equal to that quotient in the limit ok.

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$(Fg)'(c)$

$$\frac{(Fg)(c+h) - (Fg)(c)}{h}$$

$$= \frac{F(c+h)g(c+h) - F(c+h)g(c) + F(c+h)g(c) - F(c)g(c)}{h}$$

$$= \frac{F(c+h)(g(c+h)-g(c))}{h}$$

So, now that we have rewritten like this; let us write what fg should be. So, $fg'(c)$ has got to be the limit of some quotients. So, let me just write those quotients; it is got to be $\frac{fg(c+h)-fg(c)}{h}$; I want to analyze this quotient.

Well, of course we pull the standard trick that we are all familiar with, we write this as $f(c+h)g(c+h)$. So, that this is understood; $\frac{f(c+h)g(c+h)-f(c+h)g(c)+f(c+h)g(c)-f(c)g(c)}{h}$. The standard $+,-$ add something, subtract something; that gives you the answer ok.

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$$= \frac{F(c+h)(g(c+h)-g(c))}{h} + g(c)\frac{(F(c+h)-F(c))}{h}$$

Taking $\lim_{h\to 0}$ immediately gives the result.

(iii) If $g(c) \neq 0$ then

$$\left(\frac{1}{g}\right)'(c) = \frac{-1}{g^2(c)} \times g'(c).$$

Now, this will be equal to $\frac{f(c+h)(g(c+h)-g(c))}{h} + \frac{g(c)(f(c+h)-f(c))}{h}$; taking limits, limit h going to 0, immediately; gives the result. Now, we come to the next property which is the quotient property.

So, number 3; if $g(c) \neq 0$, in addition to assuming that $g'(c)$ exists; then $(\frac{1}{g})'(c) = \frac{-1}{g^2}(c)g'(c)$ ok. No assumption on the derivative at the point c, other than that it exists, but we need to assume that $g(c) \neq 0$ ok.

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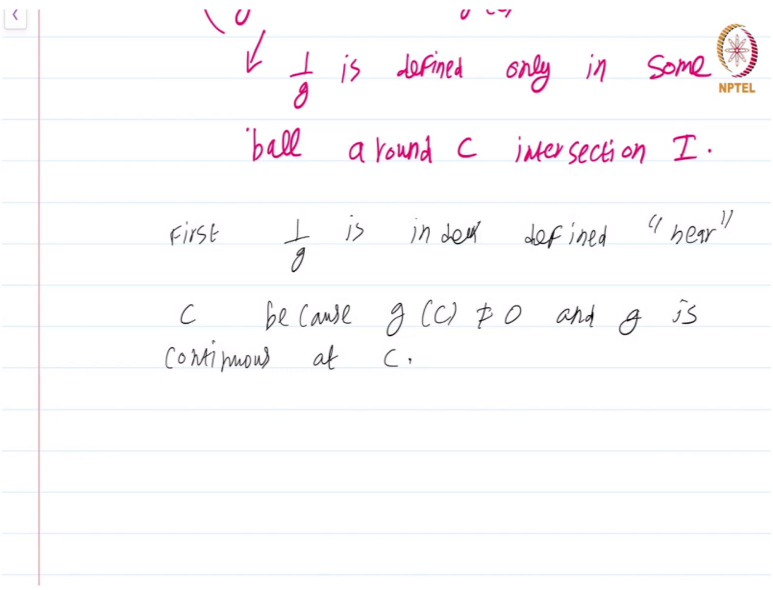

Now, note that this $\frac{1}{g}$ is defined only in some neighborhood or some ball around c, intersection I, intersection this interval I. This $\frac{1}{g}$ is not a globally defined function on the whole of I; it can happen that even though $g(c) \neq 0$, g could be 0 at some other point ok. So, let us prove this; first $\frac{1}{g}$ is indeed defined, I will just abbreviate this as saying near the point c because $g(c) \neq 0$ and g is continuous, g is continuous at c.

Because of the $\varepsilon - \delta$ definition of continuity and the fact that $g(c) \neq 0$; we can find some small enough ball around c such that the value of g on that ball intersect I; will also be not 0; this just follows from continuity and the $\varepsilon - \delta$ definition. And $\frac{1}{g}$ is indeed going to be well defined there, but we cannot say far away from this point c, whether $\frac{1}{g}$ is going to be well defined.

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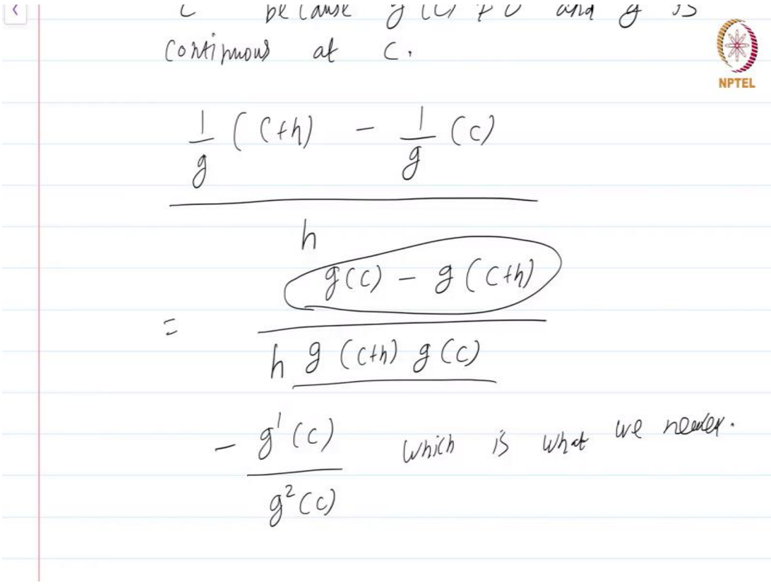

Now, let us write down what the derivative of $\frac{1}{g}$. The derivative of $\frac{1}{g}$ is got to be by writing the quotient. So, we need to compute $\frac{\frac{1}{g}(c+h)-\frac{1}{g}(c)}{h}$; we have to compute the quotient of this. Well, that is rather easy we just do LCM and this will give you $g(c+h)g(c)$.

And in the numerator, we will get $g(c) - g(c+h)$ and this h which was originally there will go here. Now, when you take limit h going to 0; you clearly get $g'(c)$ with a negative sign because there is the order has gotten mixed up. Now because of that, you get $g'(c)$ and the denominator just becomes $g^2(c)$ which is what was required; which is what we needed ok. So, the derivative of $\frac{1}{g} = \frac{-1}{g^2}g'(c)$.

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Ex: Formulate and prove a quotient rule for $\frac{F}{g}$.

Theorem (chain rule): Let $F: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ be fns. Assume

(i) $F(I) \subseteq J$

(ii) F is diff at the point $c \in I$

(iii) g ______ $F(c) \in J$.

Then $(g \circ F)'(c) = g'(F(c))\, F'(c)$.

So, exercise; formulate and prove a quotient rule for $\frac{f}{g}$, ok. Again, this is all familiar from high school; in fact, even the proofs are more or less the same. Except now that all we have done is; we have the background definition of limit rigorously done.

In school you did not have the background definition of limit rigorously done, but the proofs are exactly the same, the only difference is the certain properties of limits you took it for granted without really getting to the guts of the thing; we have already done that work. So, essentially what I am doing is a repeat of what you have done in school; so, I will be going a bit fast ok.

Now, we come to the famous chain rule; theorem, chain rule. This often perplexes students because you would expect the derivative of $g \circ f$; not to be just such a simple product. Now, let us write down; let us write down a precise statement. Let $f: I \longrightarrow R$ and $g: J \longrightarrow R$ be functions.

Assume, first that $f(I) \subset J$; this is strictly not needed you can do a better more ugly, but more refined version of this. I am not going to do that, you think about how to replace this hypothesis with the more refined one, but it will look ugly; in the after I finish the proof. f is differentiable at the point $c \in I$, g is differentiable at the point $f(c)$ in J. Then, $(g \circ f)'$ (c)= $g'\big(f(c)\big)f'(c)$, let us prove this.

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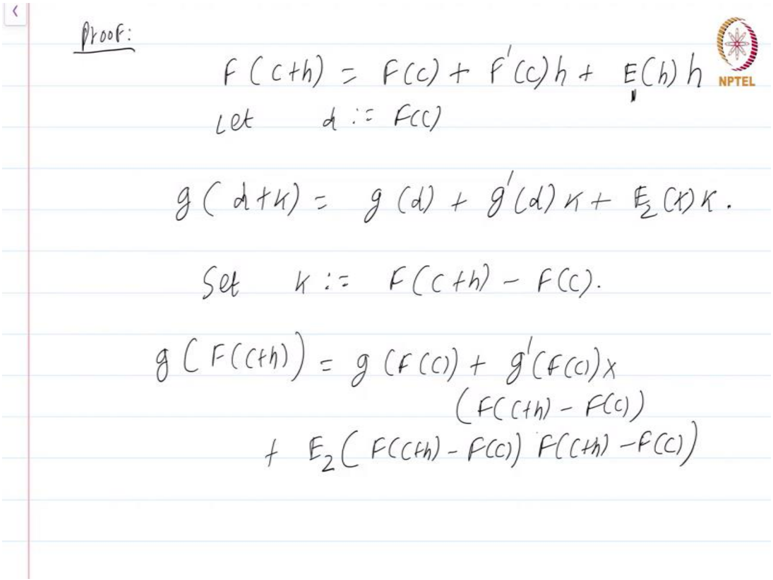

Proof, we will again use the linearization version of the definition of derivative. So, we already know that $f(c+h) = f(c) + f'(c)h + E_1(h)h$, let me call this E_1, if you do not mind. Similarly, $g(d+k)$; let $d = f(c)$. Then, $g(d+k) = g(d) + g'(d)k + E_2(k)k$.

Now, what you do is set $k = f(c+h) - f(c)$; fine. Now all this will make; so there are several underlying trivialities that as usual, I am pushing under the carpet for you to fetch. The thing is this $f(c+h) - f(c)$, this value k that you get that should ensure that $g(d+k)$ is defined ok; that should ensure that $g(d+k)$ is defined and indeed that will happen, if h is really small.

Because, then $f(c)$ will be very close to $f(c+h)$ or rather $f(c+h)$ will be very very close to $f(c)$ and all these manipulations that I am about to do will make sense. So, let us write down what this gives us $g\big(f(c+h)\big)$ because $k = f(c+h) - f(c)$ and $d = f(c)$, we get $g\big(f(c+h)\big) = g\big(f(c)\big) + g'\big(f(c)\big)\big(f(c+h) - f(c)\big) + E_2\big(f(c+h) - f(c)\big)\big(f(c+h) - f(c)\big)$ ok.

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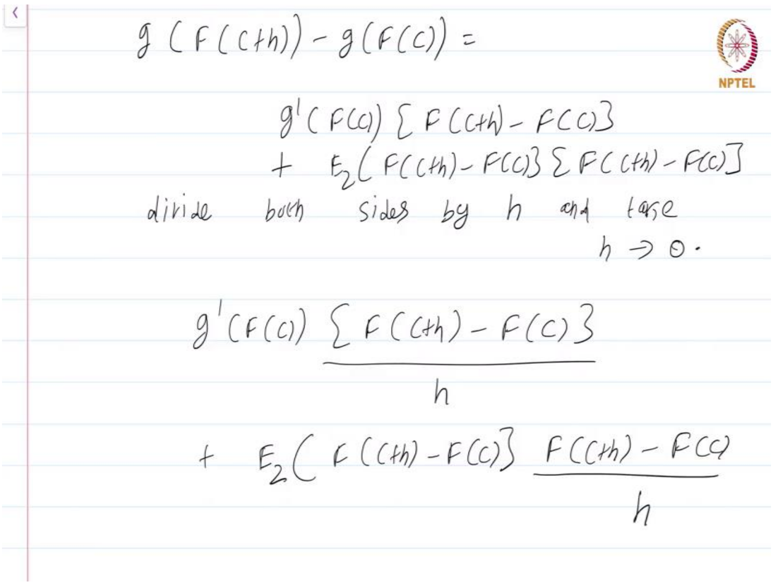

Now, what you do is; you take $g(f(c))$ to the other side. So, $g(f(c+h)) - g(f(c)) = g'(f(c))(f(c+h) - f(c)) + E_2(f(c+h) - f(c))(f(c+h) - f(c))$ ok; this is what you will get. Now, dividing both sides by h and taking h going to 0, let us see what happens.

So, divide both sides by h and take h going to 0 ok. What you will end up with is $\frac{g'(f(c))(f(c+h)-f(c))}{h} + \frac{E_2(f(c+h)-f(c))(f(c+h)-f(c))}{h}$, ok. I hope, I have not made any errors; I do not think so.

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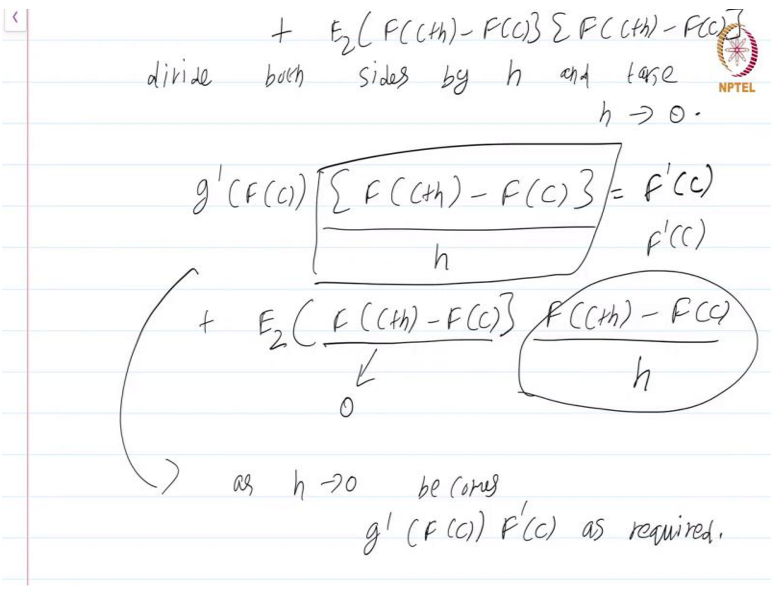

Now, as limit h goes to 0, this quantity just becomes $f'(c)$,right, and this quantity also becomes $f'(c)$. But this quantity which is there inside E_2, this will approach 0 because f is going to be continuous at the point c because f is differentiable at the point c. And we know that as the inside part of E_2 goes to 0, E_2 must also go to 0. In fact, E_2 multiplied by h, sorry; in fact, E_2 divided by h itself will go to 0 as h goes to 0. So, even without that division by h, it has to go to 0.

So, this whole thing as h goes to 0 becomes $g'(f(c))f'(c)$ as required. So, viewing derivatives in terms of linear approximations gives a more transparent proof of the chain rule, at least in my opinion. This is a course on real analysis and you have just watched the module on the basic properties of the derivative.