Real Analysis – I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 22.1
Definition and Interpretation of the Derivative

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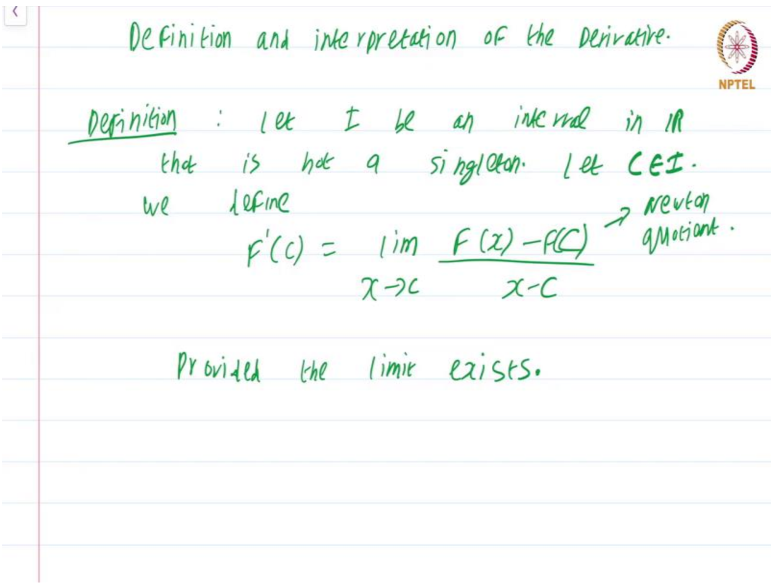

At last we are finally, here the central parts of calculus differentiation and integration. first, we go to differentiation. Without much buildup I am directly going to define the derivative because you have probably seen it once or maybe even twice.

Definition: Let I be an interval in R that is not a singleton. Let $c \in I$. We define $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, provided the limit exists, ok. So, this quantity is called the Newton quotient this $\frac{f(x)-f(c)}{x-c}$ is called the Newton quotient.

So, if you want to summarize the definition of the derivative in as quick at a time as possible you just say it is the limit of the Newton quotients. Now, you are already familiar with one particular interpretation of the derivative.

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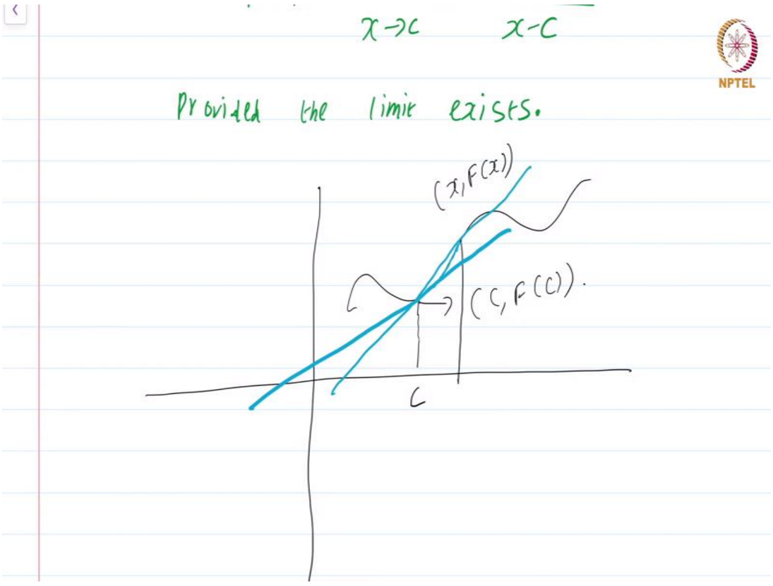

Well, let us draw a picture say you have a curve, I focus on this point c what I do is again I have a tendency of drawing these points somewhere in the air. So, this is the point c. So, essentially this will be $(c, f(c))$. If I take a point x, then this would be $(x, f(x))$, ok.

Now, observe that this $\frac{f(x)-f(c)}{x-c}$ is nothing, but the slope of this line that passes through $(x, f(x))$ and $(c, f(c))$, that is just this line ok. This does not look like a line at all. So, let me try to draw that better somewhat better not considerably ok. So, this blue line slope is given by $\frac{f(x)-f(c)}{x-c}$ that you are all familiar with from high school analytic geometry.

So, what is happening is as this x traverses along this $(x, f(x))$ traverses along this curve and approaches the point $(c, f(c))$ this line will start slowly turning and finally, at the point when you are at the limit you will sort of get this tangent line to this curve at the point $(c, f(c))$ ok. So, this is the familiar interpretation of the derivative as the limit of the secant, slopes of the secant line.

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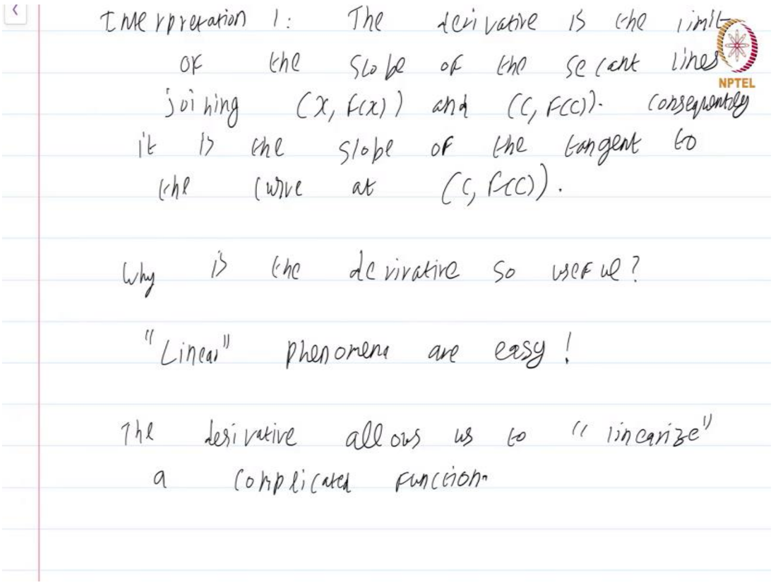

So, let me just write interpretation 1: the derivative is the limit of the secant line, the slope of the secant lines joining $(x, f(x))$ and $(c, f(c))$. consequently it is the slope of the tangent to the curve to the curve at $(c, f(c))$ ok.

So, essentially what I am doing is I am considering the function as a graph; the graph is nothing, but a curve. Then I am looking at the secant lines passing through $(c, f(c))$ and $(x, f(x))$ the line joining $(x, f(x))$ and $(c, f(c))$. Looking at the slope of those secant lines and then as you move this point $(x, f(x))$ closer and closer to $(c, f(c))$, these secant lines start moving and they become closer and closer to the tangent to this graph at the point $(c, f(c))$ finally, at the limit it will in fact, give you the slope.

This is interpretation 1 that you are very familiar with from your high school studies, but this is not in my humble opinion the best way to look at the derivative. The best way to look at the derivative is to remember what it is used for, ok. Why is the derivative so useful?

The fact that there are calculus courses in every single university in the world should tell you something that this notion of derivative is applicable everywhere. Why is the derivative so useful and this slope business does not really answer this question, right. It is a nice geometric interpretation, but it is not clear why this slope of the tangent is so useful everywhere in all endeavors of human study.

Well, let us rephrase the definition in a different way. First is this general principle that linear phenomena are easy. This is a general principle not just in real analysis or calculus, but in the whole of mathematics linear phenomena are easy ok. Linear phenomena just means something that looks like a line that is a very vague way of stating that but essentially it is not such an imprecise thing as I make it out to be.

So, linear phenomena are easy. The derivative allows us to linearize a complicated function. In fact, when you study derivatives in higher dimensions it is very common to call the derivative the linearization. Many authors do call it the linearization. What do I mean by the derivative allows us to linearize a complicated function? Well, let us look back at the definition.

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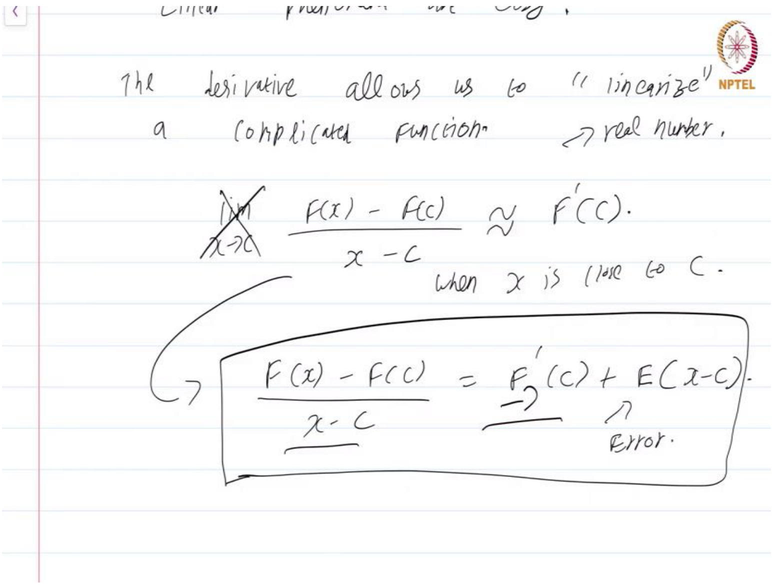

It is $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. This is nothing but $f'(c)$, right. Note that this $f'(c)$ is just a real number, it is just a real number ok. Now, what I am going to do is I am going to forget this limit. Then I cannot write equal to, this will just be approximately equal to when x is close to c right. That is what limit means as you get closer and closer to c this will become a better and better approximation.

Now, I do not know what an approximation means precisely right we are not in the habit of using sloppy language in this course. So, what I will do is instead of saying it is an approximation I will say $\frac{f(x)-f(c)}{x-c} = f'(c)$ + some error term and clearly this error term will depend on $x - c$.

It actually depends on x, but it is actually better to say it depends on $x - c$ because it sort of depends on how far away x is from c ok. So, $\frac{f(x)-f(c)}{x-c} = f'(c) + E(x-c)$, this is supposed to be the error.

Now, note this expression I could have written even if the function is not differentiable at the point c. I could have plugged in any number here I can still find an error term, right. I just take the error term to be the difference of this and this. So, I can always write this even if the function is not differentiable. When the function is differentiable, something very special happens and let us see what that is.

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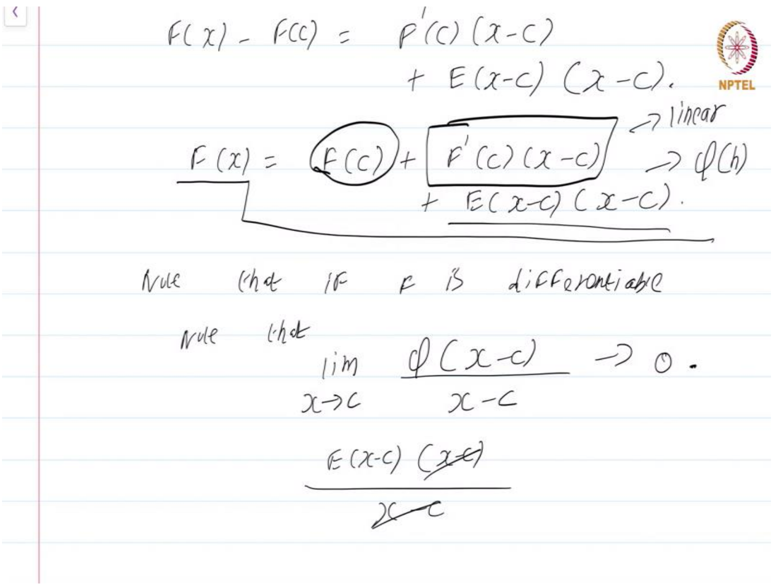

Well, let us take $x - c$ to the other side. So, what we will get is $f(x) - f(c) = f'(c)(x-c) + E(x-c)(x-c)$ ok. Now, let us do some more algebraic manipulation this is just $f(x) = f(c) + f'(c)(x-c) + E(x-c)(x-c)$.

What has happened is we have got an expression for $f(x)$, $f(x)$ is just the value at c plus this $f'(c)(x-c)$ which is certainly going to be linear. This part is linear, right, plus an error term, $E(x-c)(x-c)$, now note that if f is differentiable. So, for all the manipulations I have done does not assume that f is differentiable. In the place of $f'(c)$ I could have put any real number and I would have got the exact same expression I would have achieved nothing.

If f is differentiable, note that calling this $\phi(h)$, note that $\lim_{x \to c} \frac{\phi(x-c)}{x-c} = 0$. Well, why is that the case? Well, let us just substitute for $\phi(x-c)$, this is just $\frac{E(x-c)(x-c)}{x-c}$. $x - c$ and $x - c$ will get cancelled and obviously, as x goes to c, this error must go to 0 that is precisely the definition of saying that this $\lim \frac{f(x)-f(c)}{x-c}$ is in fact, equal to $f'(c)$ right. So, this error term goes to 0 as x approaches c. So, what does that tell you about this?

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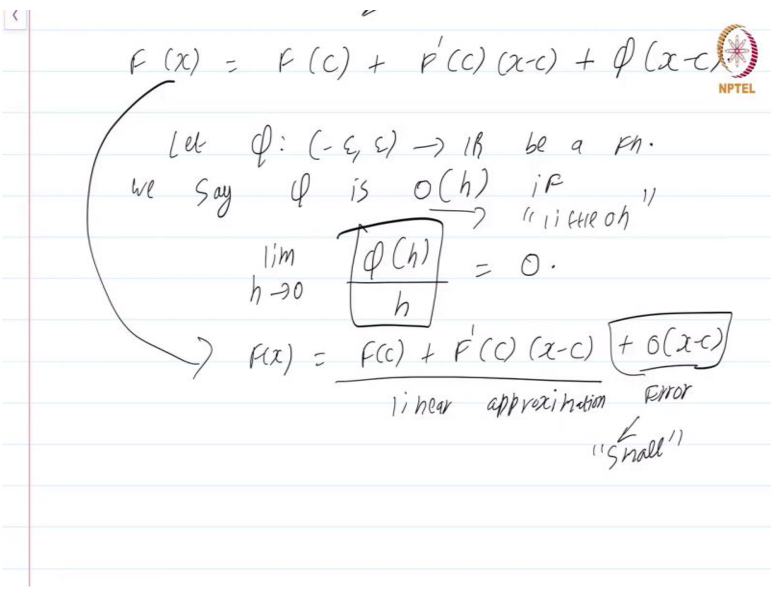

Well, we can write $f(x) = f(c) + f'(c)(x-c) + \phi(x-c)$. Let us invent a new notation rather it is already invented, by λ let us introduce a new notation that will allow us to summarize this expression that we have let $\phi: (-\varepsilon, \varepsilon) \to R$, be a function we say we say ϕ is $o(h)$ if $\lim_{h \to 0} \frac{\phi(h)}{h} = 0$ ok.

So, this little o notation if you have if you are a theoretical computer scientist familiar with computational complexity this little o notation is used extensively there it sort of measures the growth of a function. So, what this essentially says is that $f(x) = f(c) + f'(c)(x-c)$.

So, this part is the linear approximation plus little $o(x-c)$, this is the error and the fact that it is little o captures that it is small. The error is in this particular approximation is not too large that is captured by saying that it is a little $o(h)$ function ok. So, the key fact is that the function the error term is really small and that is captured by saying that when you divide by h and take the limit as h goes to 0 you still get 0.

That is simply saying that whatever this $\phi(h)$, near 0 it is even smaller than h, right, only then will $\lim_{h \to 0} \frac{\phi(h)}{h} = 0$. It will be not just smaller, it is much smaller than h. So, that when you take the quotient you still approach 0 right usually dividing by 0 leads to infinity, but that is not what is happening you are getting limit 0. So, that is sort of quantifying how small the error term is.

So, the upshot of our discussion is the derivative allows us to give a linear approximation of the function near the point c; not only does it give you a linear approximation it is a good linear approximation which is captured by saying that the error term is a little o function ok. So, this is pronounced little o. So, this second perspective is what I will focus on in this course that the derivative gives you a nice linear approximation of a function.

So, we have introduced the definition of derivative and we have seen two, one geometric interpretation and one more algebraic interpretation in terms of linear approximations; as I mentioned we will focus on this second interpretation.

This is a course on Real Analysis and you have just watched the module on the Definition and Interpretation of the Derivative.