Real Analysis - I

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Lecture – 21.3
Structure of Set of Discontinuities

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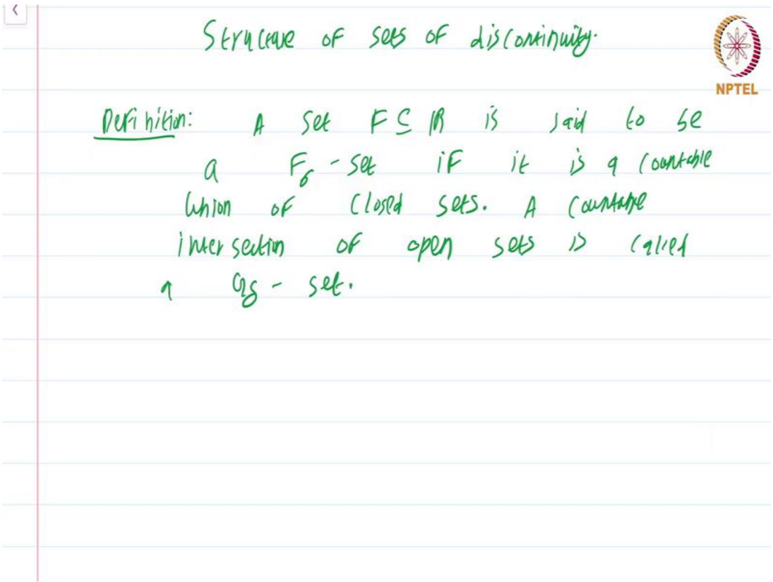

We are going to write down the structure of the set of discontinuities of a function $f: R \to R$. So, before that let me just introduce terminology which is useful. Definition a set $F \subseteq R$ is said to be F_σ set if it is a countable union of closed sets. Recall that an intersection of closed sets is always closed and a finite union of closed sets is closed.

So, countable union of closed sets need not be closed. I have given an example before. So, a countable union of a closed set is what is known as an F_σ set. A countable intersection of open sets is called a G_δ set, that the terminology is a bit weird I think it comes from the notations used in German, I am not 100 percent sure, but this is just something that has to be learnt we have to live with such terminology ok.

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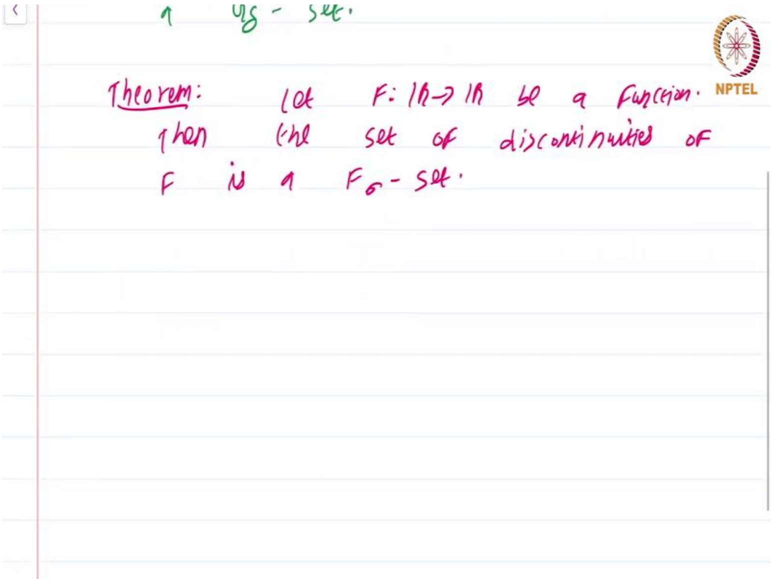

So, I can state the theorem. Theorem: Let $f: R \to R$ be a function, then the set of discontinuities of f is a F_σ set. So, you cannot have arbitrary sets has the set of discontinuity. The set of discontinuity has the precise structure as the countable union of closed sets.

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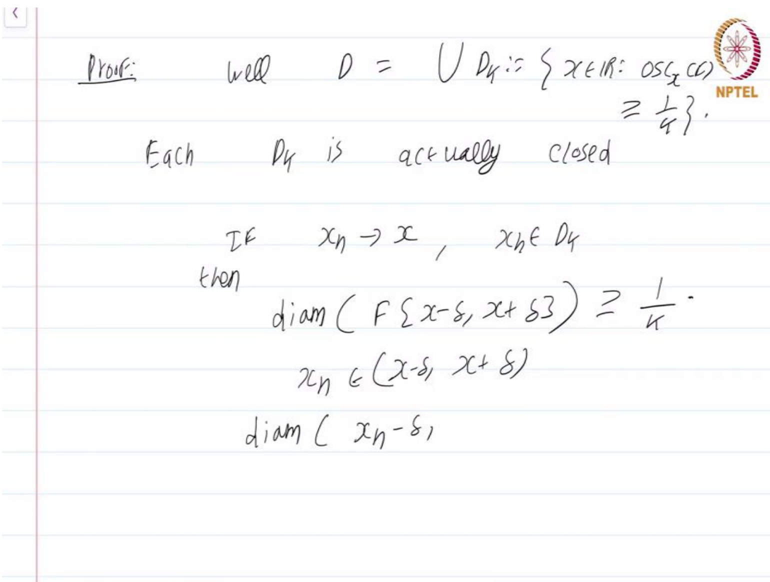

Well, the proof is rather simple. Proof well, $D = \cup D_k$, the set of discontinuities is just union of D k's, where if you recall $D_k = \{x \in R: osc_x(f) \geq \frac{1}{k}\}$, ok. Now, each D_k is actually closed; each D_k is actually closed.

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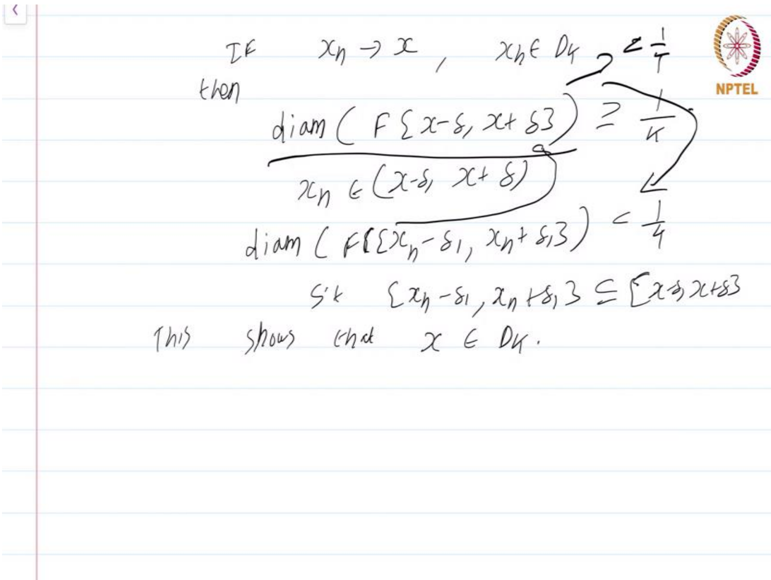

Well, why is that the case? Well, if x_n converges to x and $x_n \in D_k$, then if you look at diameter of $f([x-\delta, x+\delta])$, look at this diameter, this also has to be greater than or equal to $\frac{1}{k}$. Why is that because some x_n is an element of $(x-\delta, x+\delta)$, this open interval, some x_n is a member.

Then, if you were to choose diameter of $f([x_n-\delta_1, x_n+\delta_1])$, such that this closed interval $[x_n-\delta_1, x_n+\delta_1] \subseteq [x-\delta, x+\delta]$, if you choose it like that, then this diameter will also have to be less than $\frac{1}{k}$.

If this diameter was less than $\frac{1}{k}$, then this diameter will also have to be than $\frac{1}{k}$ simply because this is a subset of this set right.

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s.t. $\{x_n - \delta_1, x_n + \delta_1\} \subseteq \{x - \delta, x + \delta\}$

This shows that $x \in D_k$.

Each D_k is closed and we are done.

Corollary: The set of discontinuities of a fn. $f : \mathbb{R} \to \mathbb{R}$ cannot be the irrationals.

Proof: If irrationals were an F_σ-set then $\mathbb{R}$ is a countable union of closed sets. But each of the sets is nowhere set. This contradicts the baire category theorem.

So, this shows that; that x must also be in D_k, the oscillation at the point x cannot be less than $\frac{1}{k}$, if it were, then for points this x_n which is very very close to x will also have oscillation less than $\frac{1}{k}$, which is simply not allowed ok. So, this shows and in fact, that is done. I do not even have to write any more. So, each D_k is closed, and we are done. That was a rather easy proof.

All we had to observe was that these oscillations behave nicely with respect to taking limits ok. We get a nice corollary of this nice corollary. The set of discontinuities of function f from real numbers to real numbers cannot be the irrationals. You cannot have the set of discontinuities to be exactly the irrationals. Well, proof; if irrationals were an F_σ, then R is a countable union of closed sets. Why is it a countable union of closed sets?

Well, Q is a countable union of singleton sets which are closed, Q is countable. So, irrational numbers are also going to be a countable union of closed sets because it is an F_σ, we are assuming that it is an F_σ set, that is the definition of an F_σ set. But each of these sets is nowhere dense, well, that is obvious because irrationals do not contain any interval.

If you write irrationals as a union of closed sets, none of those closed sets can contain an interval; obviously, singleton sets do not contain any interval. Therefore, these are all nowhere dense sets. This contradicts Baire category theorem ok.

So, we have now shown that the set of singularity, the set of discontinuities of a function cannot be arbitrary. for instance, you cannot have the set of irrationals as the set of discontinuities.

This is a course on Real Analysis, and you have just watched the module on the Structure of Discontinuities.