Real Analysis - 1
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 21.1
Discontinuities

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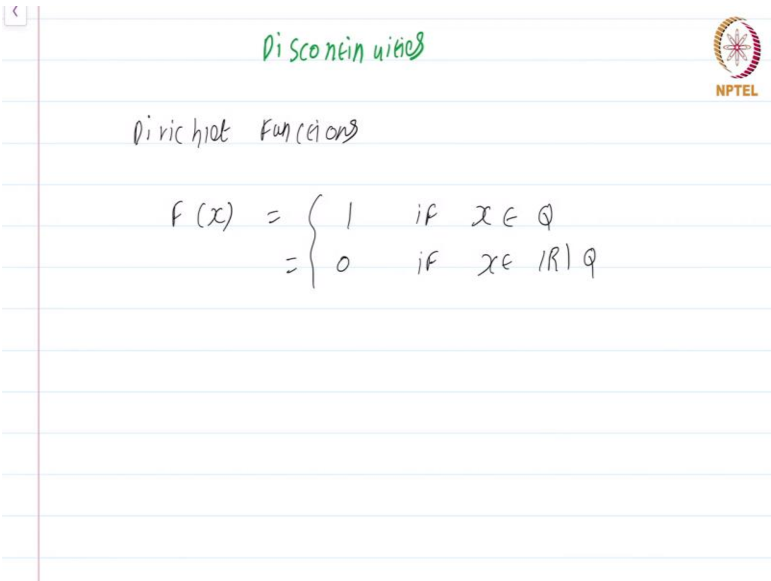

Recall the following Dirichilet function, that we had introduced a couple of weeks ago. This function was defined to be $f(x) = 1$, if $x \in Q$ and 0 if $x \in R - Q$. So, this is a function that sort of moves around between 1 and 0. Let us see what kind of discontinuities this function has.

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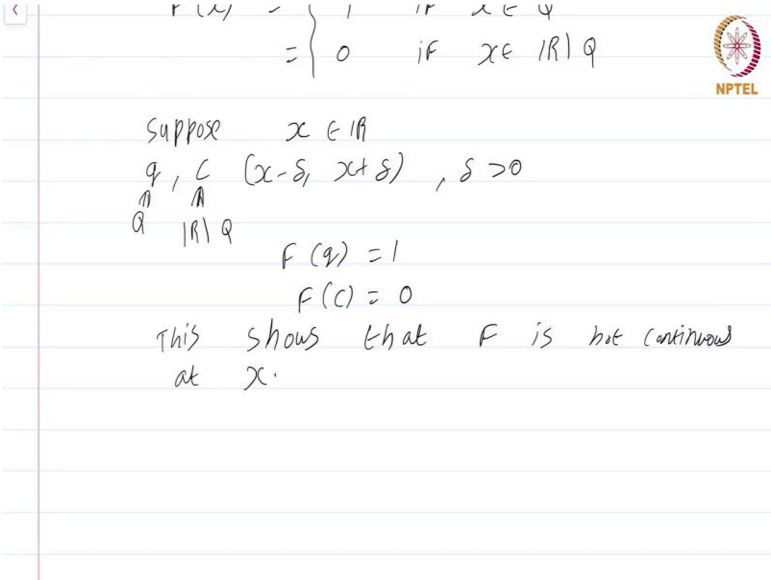

Now, suppose $x \ \in R$, it does not matter whether it is irrational or rational; then any interval $(x-\varepsilon, x+\varepsilon)$, if you take, let us not take ε, let us take δ. Any interval $(x-\delta, x+\delta)$ will contain both the point q that is rational and a point c that is irrational right.

We have already seen that Q is dense in R; so, any interval $(x-\delta, x+\delta)$, $\delta > \ 0$ will contain both the rational point as well as an irrational point. So, $f(q) = 1$ whereas, $f(c) = 0$ ok. So, this just shows that f is not continuous at x.

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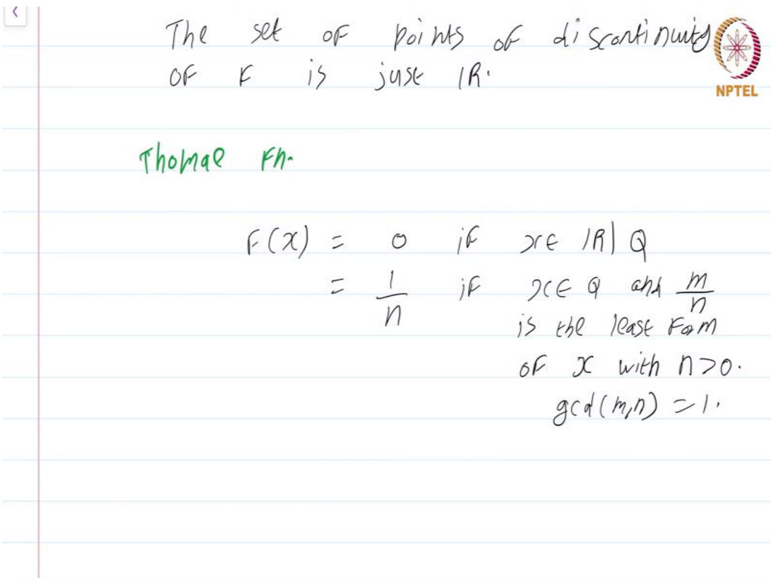

So, the set of discontinuities; so, the set of points of discontinuity of f is just R; every single point on the real line is a point of discontinuity. So, it is a pretty weird function ok. Now, let me introduce another weird function; this is called the Thomae function.

Now, the definition is somewhat similar to the Dirichilet function, but it differs at a crucial point. This is defined to be $f(x) = 0$, if x is irrational that is the same whereas, it is equal to $\frac{1}{n}$ if x is rational and $\frac{m}{n}$ is the least form of x with $n \geq 0$.

What you do is you take this rational number and write it as $\frac{m}{n}$; move if there is a negative sign, let the negative sign be stuck to the number m, cancel off all common factors. So, it is left in the least form.

So, to make this precise you can just say $gcd(m, n) = 1$. So, the greatest common divisor is just 1. So, in this scenario $\frac{m}{n}$ is said to be in its least form, just set it to be $\frac{1}{n}$ if $x \in Q$.

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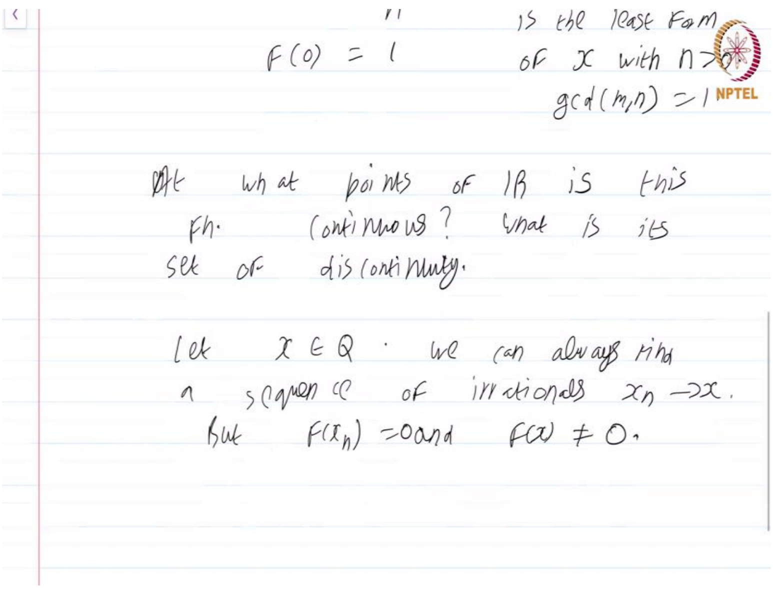

Then at what points at what points of R is this function continuous? At precisely what points is it continuous? What is its, in other words, what is its set of discontinuity? for clarity let me also define $f(0)$ precisely, $f(0)$ is taken to be 1 ok.

So, we have defined $f(x) = 0$, if $x \in R - Q$, equal to $\frac{1}{n}$ if $x \in Q$ and $\frac{m}{n}$ is the least form of x and $f(0)$ is taken to be 1 ok. Now, the question is at what points of R is this function continuous, what is its set of discontinuity?

Now, let us take $x \in Q$, ok. Then we can always find a sequence of irrationals x_n converging to x. But, $f(x_n)$ is all 0, that is the way it was defined, but and $f(x)$ is definitely not 0 right. Irrespective of what rational number you choose you either get $\frac{1}{n}$ or in the case you are at the point 0 you get 1 ok.

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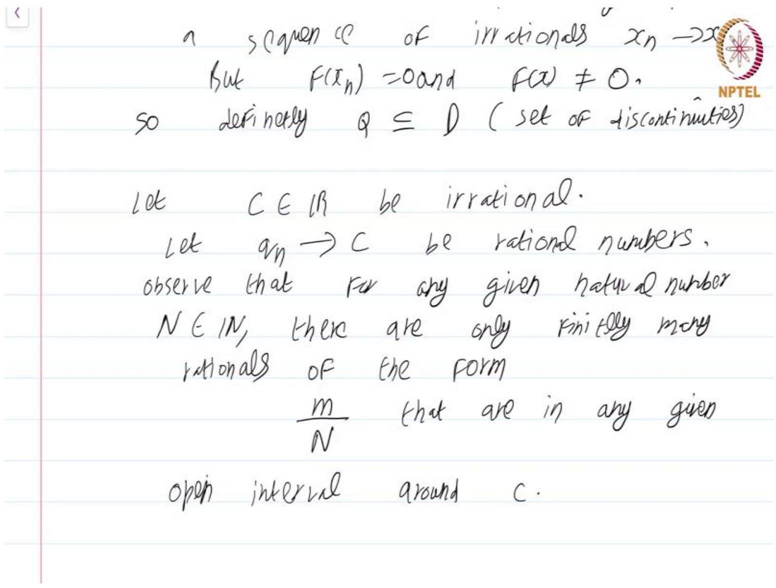

So, definitely Q is in the subset of D, set of discontinuities. So, the Thomae function definitely is not continuous at each point of the rational numbers. What about an irrational number? Here is where a fun thing happens, let us $c \in R$ be irrational ok. Now, observe what happens; let q_n converge to c be rational numbers ok.

Now, observe that for any given natural number capital N, there are only finitely many rationals of the form $\frac{m}{N}$, that are in any given neighborhood of; I will not use the word neighborhood, any given open interval around c.

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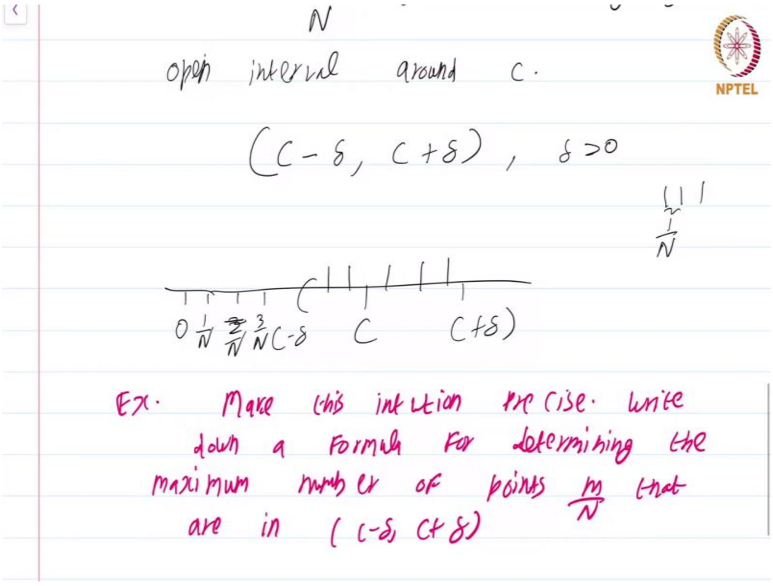

So, take this c, take $(c - \delta, c + \delta)$, take this open interval, $\delta > 0$, ok. So, here you have the point c, here you have $(c - \delta)$, here you have $(c + \delta)$.

What this is saying is there are only finitely many rationals of the form $\frac{m}{N}$ that are in any given open interval around c. Why is this? Well, think about this; imagine you have put a ruler on the real line. The rulers are the rather a scale whose rulings are all separated by $\frac{1}{N}$, ok.

Imagine you have put a ruler on the real line separated by $\frac{1}{N}$. So, if the origin is here you have $\frac{1}{N}$, you have $\frac{2}{N}$, you have $\frac{3}{N}$, so on ok. So, it is obvious that only finitely many such markings will be contained in $(c - \delta,\ c + \delta)$ that is obvious. So, and these will correspond, these will correspond to the various numbers of the form $\frac{m}{N}$ that are in any given, that are in this interval $(c - \delta,\ c + \delta)$.

So, I leave it as an easy exercise to make this intuition precise; exercise: make this intuition precise. In fact, a better exercises: write down a formula for determining the maximum number of points $\frac{m}{N}$ that are in $(c - \delta, c + \delta)$. In fact, you can write a formula that tells you the maximum number of numbers of the form $\frac{m}{N}$, that are in $(c - \delta, c + \delta)$ ok.

Regardless of whether you are convinced by this ruling argument or if you want to make this precise, it is up to you. Regardless, I am going to take it that you understand that in this interval $(c-\delta, c+\delta)$ there are only finitely many rationals $\frac{m}{N}$.

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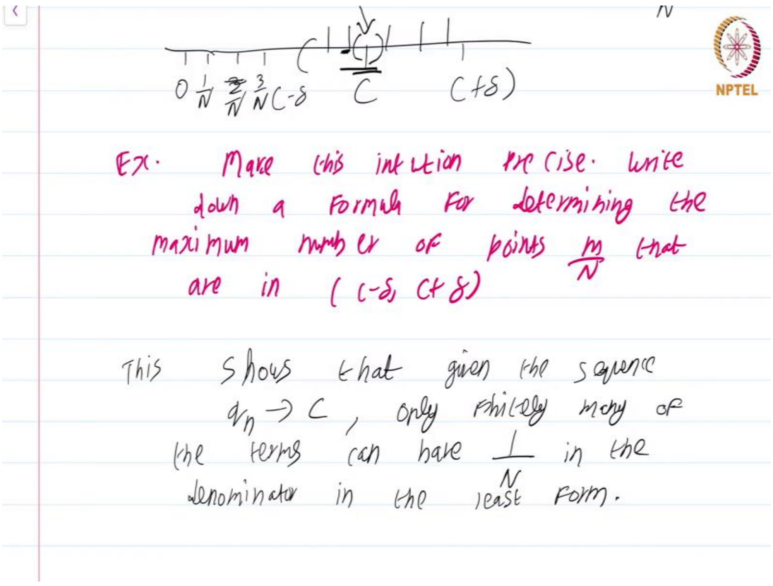

What does this show? Well, this shows that given the sequence q_n converging to c, only finitely many of the terms can have $\frac{1}{N}$ in the denominator in the least form.

Well, why is this the case? Well, again you can expand it and give a mathematically precise reason, but the reason is as follows. Given any open interval $(c-\delta, c+\delta)$, eventually the terms of the sequence will have to be in this interval. And, we already know that there are only finitely many such points of the form $\frac{m}{N}$ in this interval.

So, if the very last point closest to c is here, then if I choose a much smaller δ and I choose this new interval, then none of the points of the form $\frac{m}{N}$ are going to be in this new interval ok. Note, I am crucially using the fact that c is irrational; that means, the markings of the ruler can never coincide with c, they will have to be, they can be near c, but they cannot be arbitrarily close to c.

So, I can always choose a much smaller interval; let us say $\left(c-\frac{\delta}{10{,}000}, c+\frac{\delta}{10{,}000}\right)$ such that none of these points of the form $\frac{m}{N}$ are in this new interval. But, the sequence q_n converges to

c; so, q_n must eventually be in this newer open interval and here there are none. So, after a point all the terms q_n cannot have the form $\frac{1}{N}$ in the denominator.

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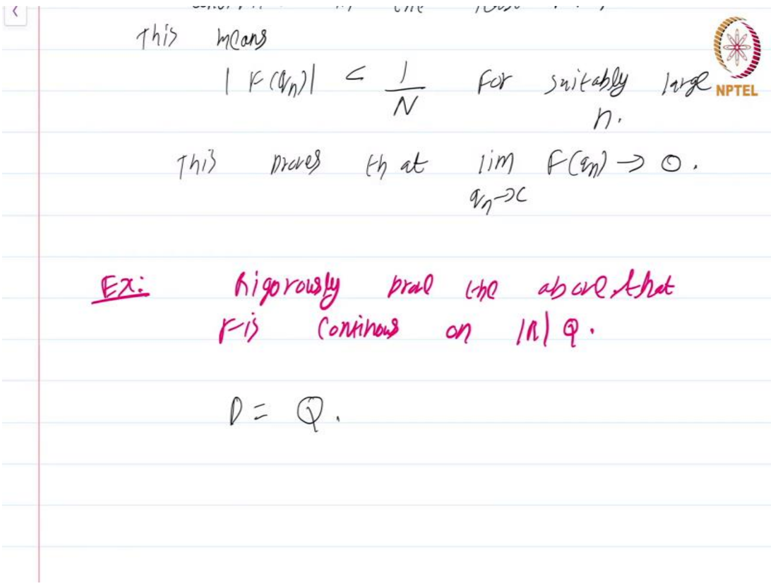

So, this means $|q_n| < \frac{1}{N}$ for suitably large small n ok. Why is this? Well, you have to think about this for a moment, you just look at $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N}$; there are only finitely many rational numbers which have $1, 2, 3, 4, \ldots, N$ in the denominator.

If you choose a small enough neighbourhood around this point c, then none of those rational numbers can be in this small enough neighbourhood. Then what happens is if q_n is suitably large then q_n's are all going to be in this small neighbourhood around this point c.

Then none of them can have the denominator of the form $\frac{1}{N}, \frac{1}{N-1}, \ldots, 1$; that means, the denominator has to be greater than N. That means, $|f(q_n)| < \frac{1}{N}$ for suitably large n. This proves that $lim_{q_n \to c}\, f(q_n) = 0$ ok.

So, I have been very very intuitive here, a lot of things I have explained verbally without writing down. So, that is all a ploy of mine to get you to solve something non-trivial on your own. Rigorously prove the above ok. What have we shown? We have shown that if you take any rational sequence q_n converging to c, then $f(q_n)$ must go to 0.

Obviously, if you take an irrational sequence converging to c, f of that sequence; obviously, converges to 0; they will all be 0. Putting this together, f is actually continuous at the point c.

So, rigorously prove the above f is continuous on $R - Q$. So, I want you to write down a rigorous proof of this making all my verbal arguments mathematically precise. It is not hard; the idea is the hard part, converting it to the mathematical proof is not that hard ok.

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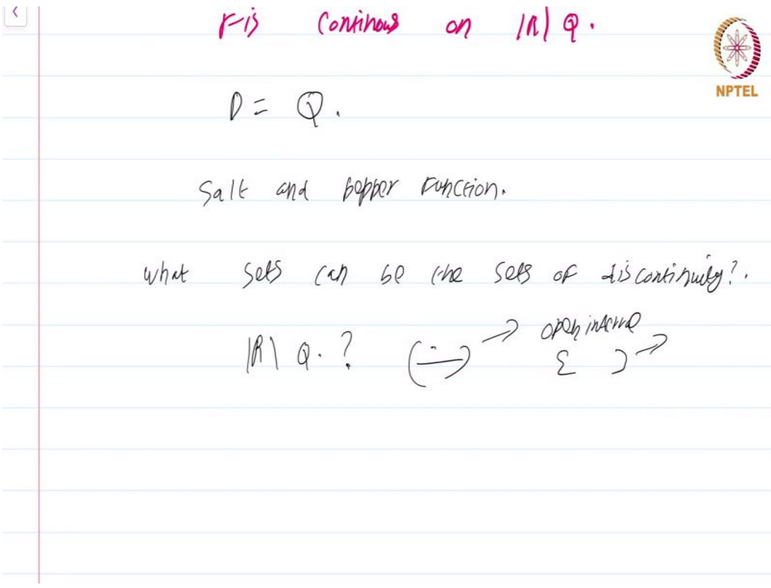

What does this show? This shows that D, the set of discontinuities is Q. So, we have got some interesting stuff, we have found a function which is very weird; by the way this function is also called the salt and pepper function; that is a nice terminology. It is called the salt and pepper function. So, this function, the Thomae function has set of discontinuities precisely the set Q.

So, the next question is what sets can be the sets of discontinuity? Precisely, what sets can be the sets of discontinuity? It is very easy to see that any finite set can be the set of discontinuity, that is I leave it to you as a very simple exercise. And, we have just seen that the complicated set Q can be the set of discontinuities.

We have seen that the whole set R can be the set of discontinuities. What about some set like $R - Q$, can this be the set of discontinuities? Can an open interval be a set of discontinuity, open interval? Can a closed set, closed interval be set of discontinuity? Well, these are questions that have to be answered. So, but before that, let us give a definition that allows us to study discontinuities in the first place.

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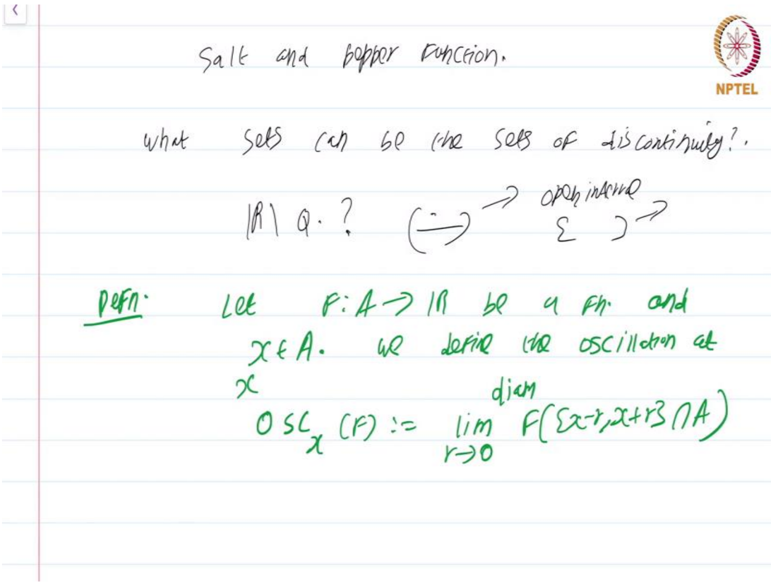

Definition: let $f : A \to R$ be a function and $x \in A$. We want to see what happens if f is not discontinuous at the point A. We want to somehow quantify the discontinuity. How we do that is we define the $osc_x(f)$. This is denoted oscillation at x of f to be just the $\lim_{r \to 0} diam\, f\left((x-r, x+r) \cap A\right)$ ok; limit of diameter of this set.

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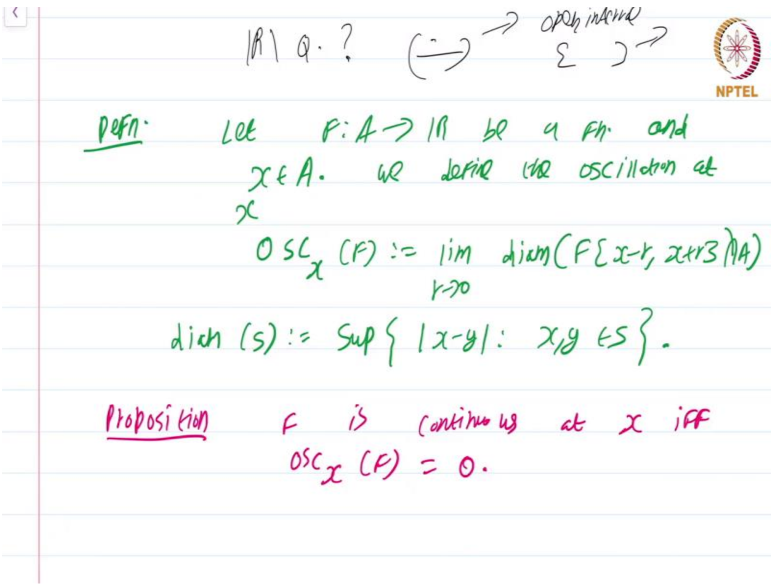

So, let me just write that down precisely. This is $\lim_{r \to 0} diam\, f\left((x-r, x+r) \cap A\right)$. Now, I must tell you what this diameter is right, that was not something that we have studied so far.

Well, that is nothing, the diameter of a set S is just the largest possible distance between two points of S. How do we capture that? Well, the $diam\ (S)$ = su p$\{\ |x-y|: x, y \in S\}$, ok.

If you think about this, it will turn out that the diameter of a circle in R^2; though we have just defined it for R, you can think of similar definitions for R^2. The diameter of a circle in R^2 will just turn out to be the usual length of the diameter ok. If you take any set S, you look at various points $x\ , y$ coming from S, look at the distances, look at a maximum possible distance that is called the diameter.

Now, what is this $osc_x(f)$ measuring? Well, it is measuring how large is the image of f near the point x. If the function is not continuous then you expect the function to have a non-zero oscillation at the point x and that is indeed the case that is indeed the case. So, I am going to leave it to you to prove this very easy proposition, very easy proposition.

If f or rather let me just phrase it this way, let me just f is continuous at x if and only if the $osc_x(f) = 0$. So, if the oscillation is 0 then the function f is continuous, if the function is continuous then the oscillation is 0. So, the oscillation measures whether the function is continuous or not, but it also quantifies how far away it is from being continuous at the point x ok.

In the next module, we shall use this oscillation to study the discontinuities of a monotone function; recall monotone functions are those that are either increasing or decreasing. And, we will then use this oscillation function to measure or rather to prove what kind of states can be the set of discontinuities.

The answer is surprisingly not all the subsets of R. The subsets of R that are allowed to be discontinuities are special and we will exactly see what they are. And the oscillation, a measure of how far a function is away from being continuous is a key tool. This is a course on real analysis, and you have just watched the module on discontinuities.