Real Analysis - I
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Lecture – 20.3
The Baire Category Theorem

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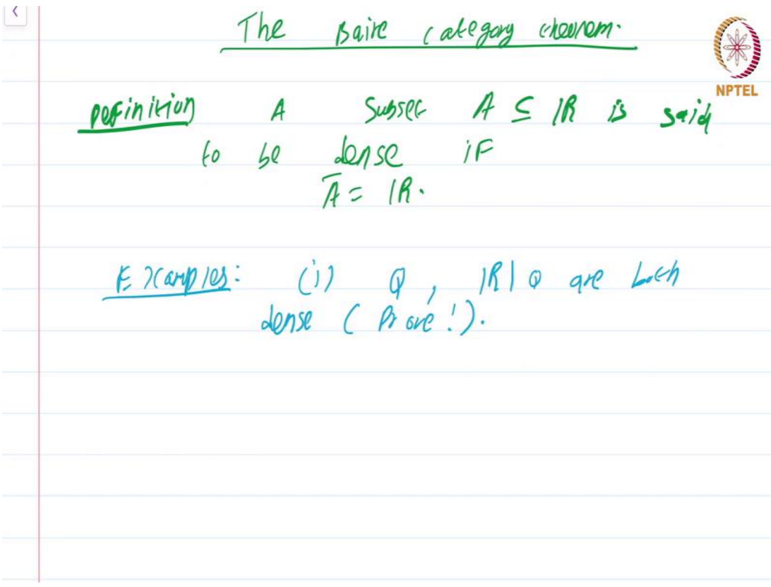

In this module we are going to prove a somewhat technical result called the Baire Category Theorem. We will use this result in studying the structure of the discontinuities of a function, but the deeper applications of this result can be only seen in a future course on functional analysis. So, without further ado let me just state a definition which is long pending definition. A subset $A \subset R$ is said to be dense if $\overline{A} = R$.

Immediately we have examples way back from week 2 that Q, the rational numbers and $R - Q$ are both dense, well, proof. What we call density way back in week 2 was slightly different from this definition, but you will be able to show quite easily that both Q and $R - Q$ are dense. So, let me state the Baire category theorem and prove it.

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Baire Category theorem: Let $G_n \subseteq \mathbb{R}$ be a sequence of open and dense subsets. Then

$$G := \bigcap G_n \text{ is also dense.}$$

Proof: Let $x_0 \in \mathbb{R}$ and let $I_0 \subseteq \mathbb{R}$ be any closed non-singleton interval that contains x_0.

Because G_1 is dense in $\mathbb{R}$

$\overline{G_1} = \mathbb{R}$. There is a sequence $y_n \in G_1$

$y_n \to x_0$. This means that for large n, $y_n \in I_0$.

The proof is not too hard in fact, the proof that I am about to give is very similar to the proof of the uncountability of R that we saw using the nested intervals theorem. Let $G_n \subset R$ be a sequence of open and dense subsets. Then G defined to be $G = \cap G_n$ is also dense. So, if you take a bunch of open and dense subsets then the intersection is also going to be dense.

Proof and after going through this proof look back at the proof of the uncountability of R using the nested intervals theorem that we saw and look for the similarities. So, let $x_0 \in R$ and let $I_0 \subset R$ be any closed non-singleton interval that contains x_0 ok.

Now, because G_1 is dense in R, $\overline{G_1} = R$, this is the very definition. That means, there is a sequence $x_n \in G_1$, let me not use x_n you will understand in a moment why, there is a sequence $y_n \in G_1$ such that y_n converges to x_0, that is because that is the definition of closure ok. This means that for large n, y_n is an element of I_0, right, because the sequence y_n converges to x_0 for suitably large n, y_n must be in this closed interval I_0, ok.

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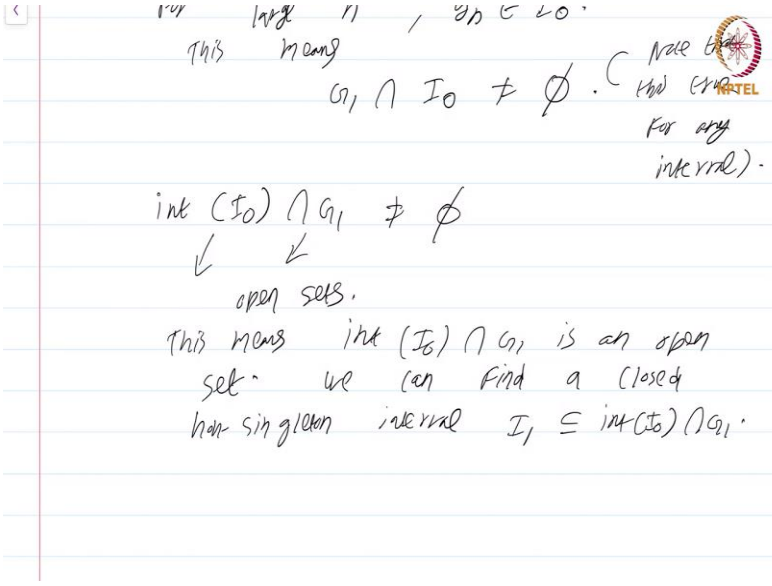

This means $G_1 \cap I_0 \neq \phi$, ok. In fact, note that this is true for any interval note that this is true; this is true for any interval ok. This is just a side remark what we have shown is if you take a dense set then intersection of the dense set with any non singleton interval will definitely be a non empty set ok. Now, how is this useful, well look at interior of I_0; look at interior of $I_0 \cap G_1$, by a similar argument this is also going to be non empty ok.

Essentially, what we used in the previous argument is the fact that for suitably large n, y_n will belong to any given interval of x_0. So, in particular, this taking it to be an open interval gives no additional difficulty ok.

But both of these are open sets ok, this means interior $I_0 \cap G_1$ is an open set because intersection of two open sets is an open set ok. Now what we do is, we can find a closed interval, closed non singleton interval I_1 that is contained in interior of $I_0 \cap G_1$.

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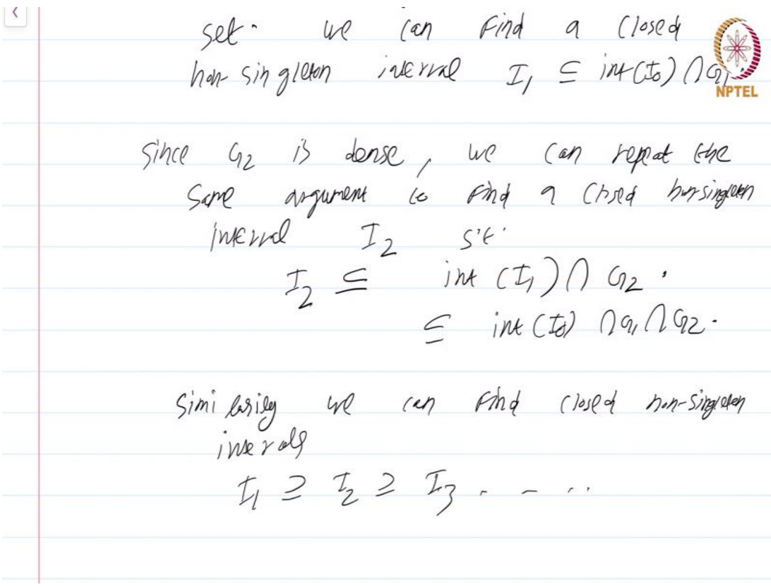

Now, since G_2 is dense by exactly the same argument, notably what is there in the parenthesis here since G_2 is dense we can repeat the same argument; repeat the same argument to find to find a closed non-singleton interval I_2 such that I_2 is contained in interior of $I_1 \cap G_2$, ok.

In fact, if you think about this carefully because I_1 is actually contained in interior of $I_0 \cap G_1$ this I_2 is also going to be a subset of interior of $I_0 \cap G_1 \cap G_2$ ok. Similarly, we can find closed non singleton intervals non-singleton intervals I_1, containing I_2, containing I_3, ... such that.

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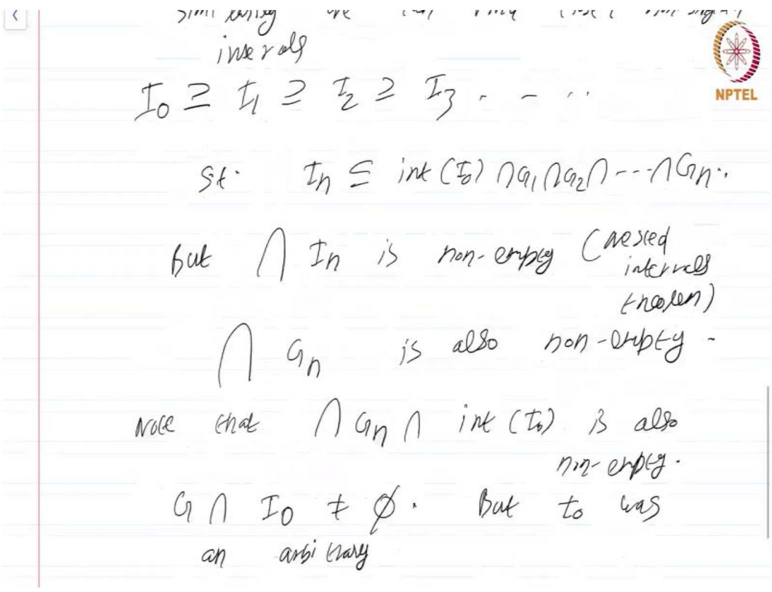

I_n is contained in interior of $I_0 \cap G_1 \cap G_2 \ldots \cap G_n$ ok Excellent, but intersection of $\cap I_n \neq \phi$ by nested intervals theorem ok. This just means that the $\cap G_n \neq \phi$, ok.

Now, how does this help us, well note that intersection $\cap\, G_n \cap \text{int}\, I_0 \neq \phi$, simply because each one of the intervals is contained in I_0, that is how we had constructed these.

So, what we have shown is the set G intersects I_0, ok, but I_0 was an arbitrary interval, right, it was just an arbitrary interval, we did not put no hypothesis on I_0. I just said let $x_0 \in R$ and let $I_0 \subset R$ be any closed non singleton interval that contains x_0 that is it. So, but I_0 as an arbitrary interval that contained the fixed point x_0.

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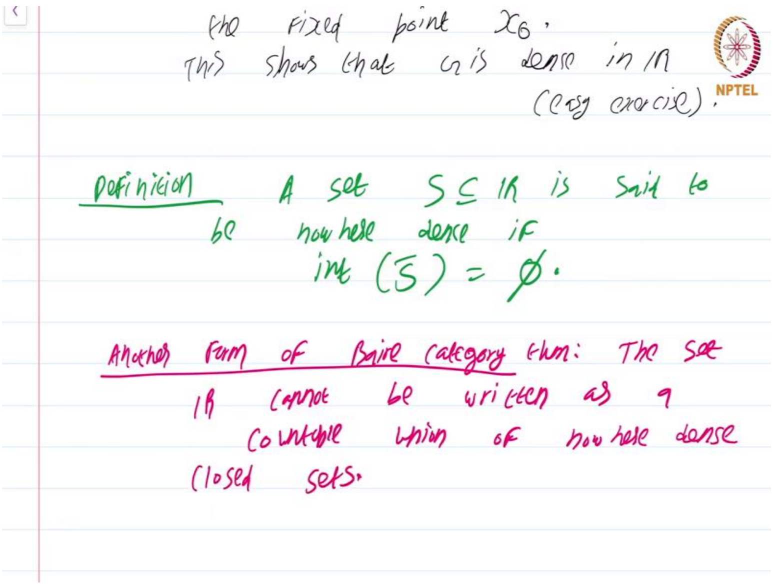

Now, what does this show us? This shows that no matter what point x_0 you take and whatever small interval around that point you take there is a element of G that is in that interval. So, this shows that G is dense in R ok, I have done most of the work just from concluding that G is dense in R is an easy exercise ok. So, this concludes the proof of the Baire category theorem.

Now, usually the Baire category theorem is stated in a somewhat different way, let me just state it, but I will not elaborate on it because it starts to get a bit technical. A set $S \subset R$ is said to be nowhere dense is said to be nowhere dense if $\text{int}\, \overline{S} = \phi$ ok.

So, we say that a set is nowhere dense if at no point does it accumulate an entire interval whereas, a dense set is something that accumulates all intervals. This is something that does not accumulate any interval, the terminology is a bit weird, but you will get used to it when you are in a more advanced course.

So, there is another form of Baire category theorem. Another form of Baire category theorem that states the following. The set R cannot be written as a countable union of nowhere dense closed sets and the proof is left as an easy exercise for you.

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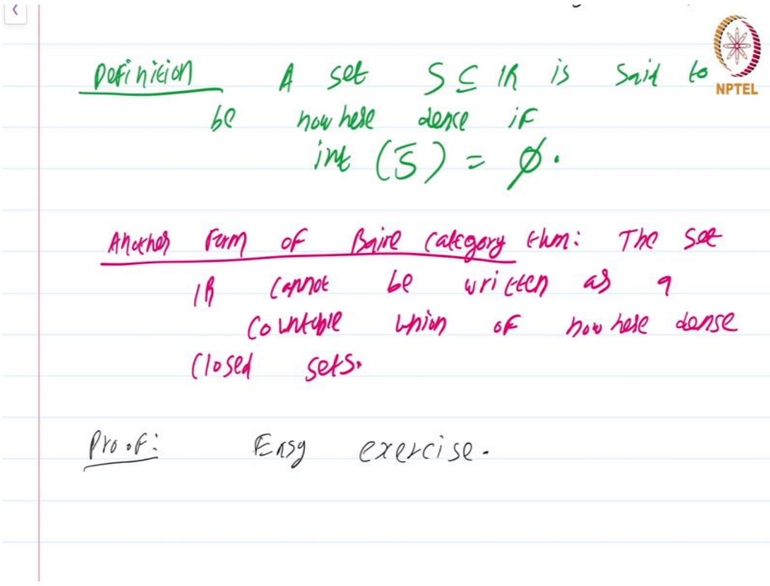

This will be just translating terminology and using the previous version of Baire category theorem. So, proof is easy exercise. So, I am being a bit quick because this is in fact, a theorem that is best thought in a later course, but nevertheless we need it as a tool in one of the theorems to follow. So, please do not focus too much on this theorem you there is a time and a place where you will get a chance to understand the intricacies of the Baire category theorem.

This is a course on real analysis and you have just watched the module on the Baire category theorem.