Real Analysis - I
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Lecture – 20.2
The Structure of Open Sets

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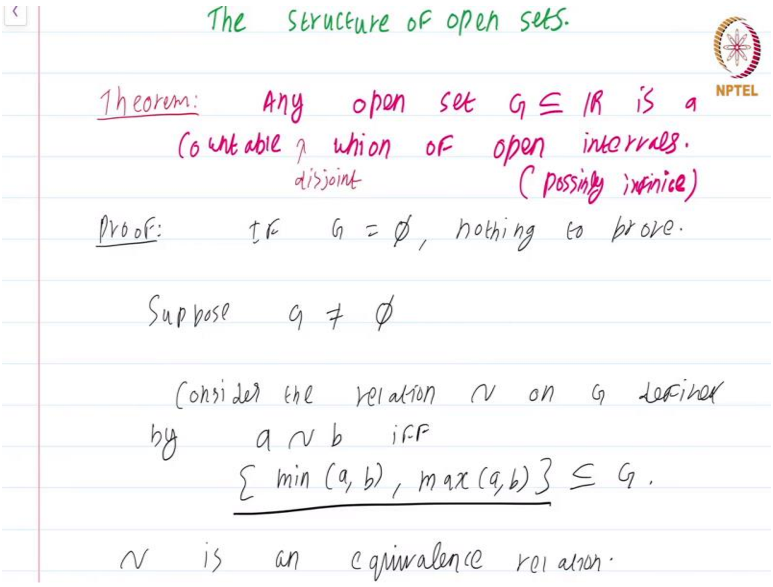

All open intervals are also open sets thankfully. How complicated can an arbitrary open set look like? In this module we shall answer this question. I will directly state the theorem. Theorem: Any open set $G \subseteq R$ is a countable union of open intervals; here countable of course includes the possibility of finite.

Proof: First if $G = \phi$, nothing to prove because the empty set is trivially an interval by our definition of an open interval and there is nothing to prove. G is just the union of just one open interval which is an empty set.

Suppose $G \neq \phi$. We have to show where G is a union of open intervals and there are only countable many such open intervals. How do we do this? Well, consider the relation $\sim$ on G defined by $a \sim b$ if and only if $[\min(a, b), \max(a, b)] \subseteq G$.

So, consider two points a, b look at their minimums, look at their maximums and look at the closed interval created by the minimum and maximum; then $a \sim b$ if and only if $[\min(a, b), \max(a, b)] \subseteq G$. Now, as you can guess what I am going to prove is that this is an equivalence relation.

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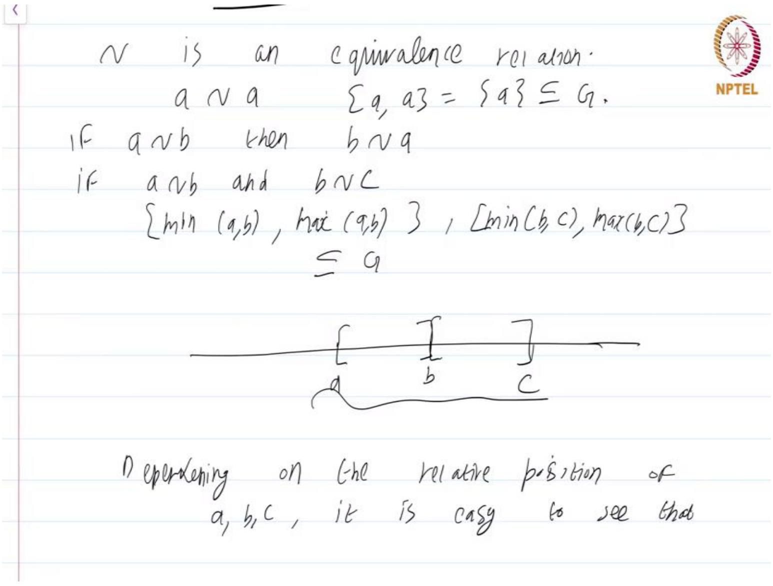

$\sim$ is an equivalence relation. How do we show this? Well, a is clearly related to a because the closed interval $[a, a] = \{a\}$, which is obviously going to be a subset of G.

So, reflexivity is obvious. Similarly, if $a{\sim}b$, then obviously $b{\sim}a$, right because this $\min(a, b)$, $\max(a, b)$ will be the same. Now, suppose if $a{\sim}b$ and $b{\sim}c$, then note that $[min(a, b), \max(a, b)]$, $[min(b, c),\ max(b, c)] \subseteq G$ ok. Now, this is best proved by just giving a simple picture.

We can assume that a is like this, b is like this, c is like this, that is one case. Well, we already know that this closed interval is a subset of G and this closed interval is also a subset of G. So, consequently this subset $[a, c] \subseteq G$ also because it is just the union of the closed interval $[a, b]$ and closed interval $[b, c]$.

Now, similarly there are other possible configurations of a, b, c and in each one of those you can trivially see that it will always be the case that $[\min(a, c), \max(a, c)]$ is always going to be a subset of G.

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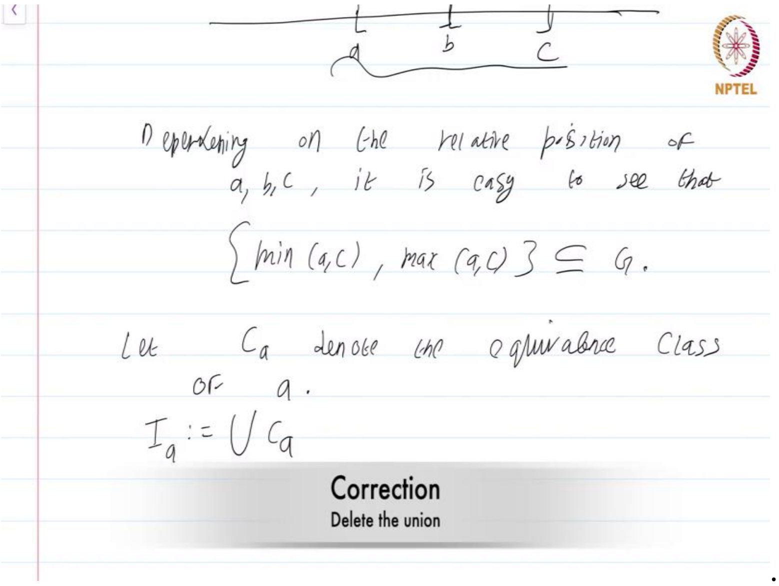

So, depending on the relative positions of a, b, c; a, b, c it is easy to see it is easy to see that $[\min(a,c)\,,\max(a,c)]$, closed interval is a subset of G ok.

Please finish this argument it is rather easy. So, now, that we have an equivalence relation, let C_a denote the equivalence class of a ok. Now, what will happen what will happen if you take union of C_a? What will happen if you take union of C_a? Well, I just call this I_a. I define I_a to be union of C_a, ok.

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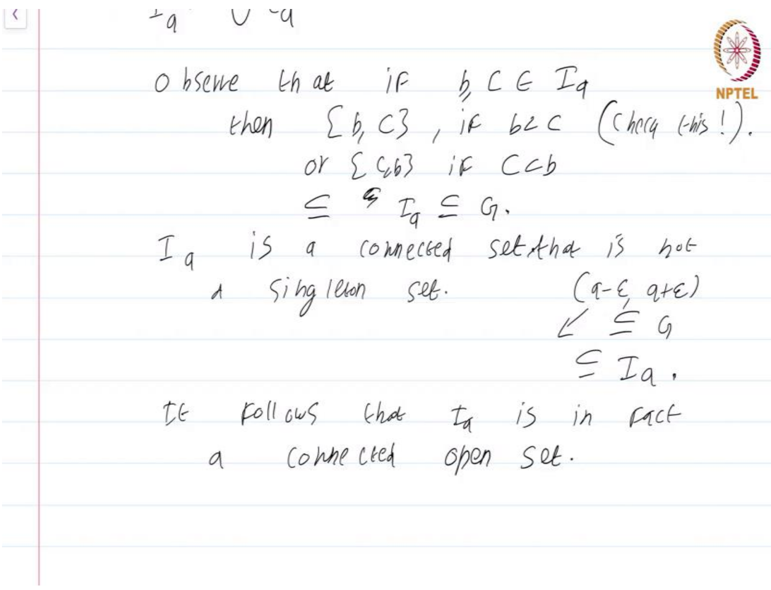

First observe that if b, c are elements of I_a, then this interval $[b,c]$, if $b < c$ or $[c,b]$, if $c < b$, both are, I mean not both, one of them is a subset of G right.

In fact, one of them would be, I will not say subset of G that is not clear, will be a subset of I_a therefore, a subset of G. Why will it be subset of I_a? Simply because look at the definition of this equivalence relation. It says that the minimum of the two, maximum of the two should be a subset of G.

Therefore, if you take these points $b,c \in I_a$; that means, the if you look at the min$\{b,c\}$, max$\{b,c\}$ should also be in I_a and that will just follow from transitivity when you apply it with the third element a ok. So, in any case please check this; this is also fairly easy to see, fine.

So, what has happened is I_a is a connected set. ok. Not only that I_a is a connected set, it is a connected set that is not a singleton set. Why is that because, this set G is open. So, $(a-\varepsilon, a+\varepsilon)$ will be a subset of G and you can clearly see that this will also have to be a subset of I_a ok, where this ε is suitably small.

Now, because of this fact that $(a-\varepsilon, a+\varepsilon)$ will be a subset of G and therefore, a subset of I_a it follows that I_a is a connected open set fine.

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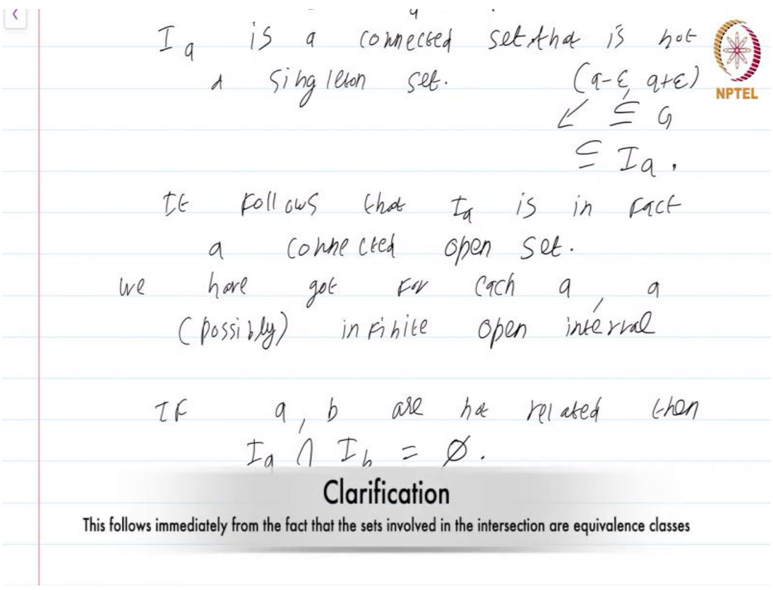

So, what have we got? What we have got is the following. We have got for each a for each a, a possibly infinite open interval ok.

So, to be 100 percent precise, I must say countable union of open interval,s possibly infinite. I am allowing intervals where one or both endpoints could be infinity. So, you could have the open interval $(-\infty, c)$ or the open interval (c, ∞) those are allowed in this theorem ok.

So, coming back to the proof, what we have is for each a, possibly infinite open interval I_a. Now, if a, b are not related, then $I_a \cap I_b = \phi$, ok.

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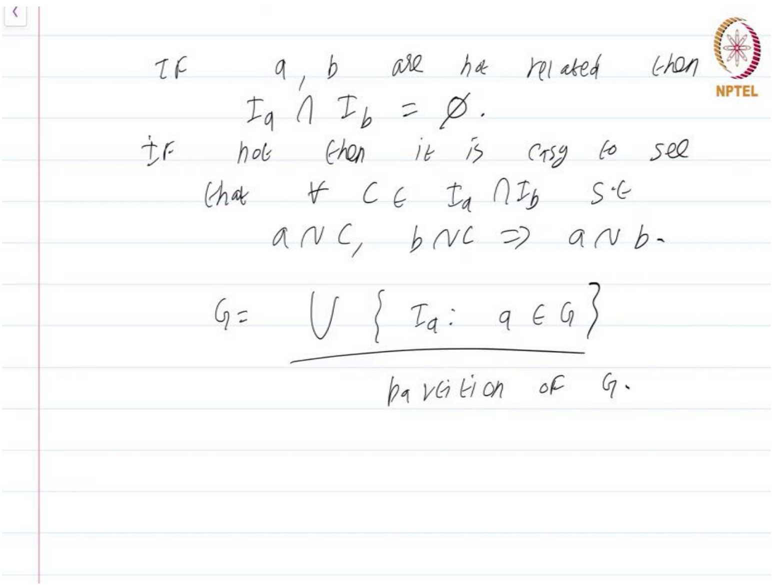

Why is this the case? Well, if not; then it is easy to see that there exist $c \in I_a \cap I_b$ such that $a \sim c$ and $b \sim c$ which implies $a \sim b$, ok.

So, if it were the case that $I_a \cap I_b \neq \phi$, then in fact, I need not write there exist, then for all. Then if you take some $c \in I_a \cap I_b$, then that c will be related to both a and b, therefore, a and b are related which is not possible ok. So, what we have managed to do is we have shown that G is a disjoint union of the I_a's, which I will denote by the square cup, disjoint union of various I_a's ok.

So, what I am doing is, I am essentially going to pick one element G and look at the equivalence class C_a and take the union and get I_a, then take another element b which is not related to a then I will get another interval. In this way I can exhaust G as a disjoint union of open intervals ok. So, better to actually say, G is the union of the collections I_a, such that I_a, a comes from G and this is a partition of G ok.

So, any two elements in this collection I_a such that a comes from G will either be disjoint or will be equal. So, therefore, this will form a partition. So, we have managed to write G as a countable union not a countable union as a disjoint union of intervals. So, in fact, we have got some things stronger than what we have stated. So, let us put that; is a countable disjoint, disjoint union of open intervals, possibly infinite.

So, we have strengthened. In fact, that is what I wanted to set out to prove, I just forgot mentioning that this is going to be a disjoint union ok. So, now, we have got G as the union, why is it countable? Well, I am going to produce a list for you.

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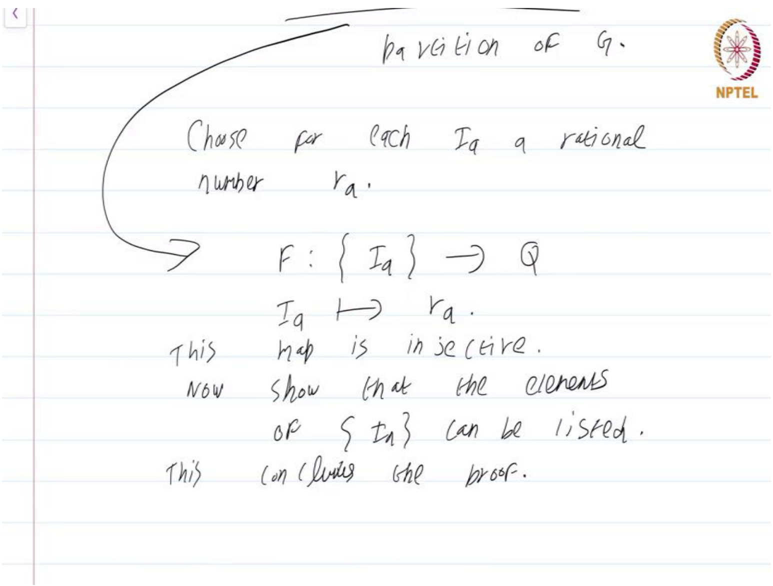

How am I going to produce a list? Choose for each I_a, a rational number r_a ok. Then all the different members there are, many of them that coincide simply because this is a partition.

So, if $I_a \cap I_b \neq \phi$, then $I_a = I_b$. What I do is I just choose one interval. I mean one I_a if there is a different I_a then I put it there. If there is another I_a, that is the same, then I discard it. So, that means, I am just looking at although distinct members of this, I can just write it as, I can consider a map $f: \{I_a\} \to Q$.

And that is given by this I_a gets mapped to r_a ok, and this map is injective. This map is injective because in this collection I_a, I am explicitly discarding the repeated members ok. Now, show that the elements of this collection I_a can be listed.

Once you have a set and an injective mapping to a countable set, then that set is countable. You have already solved this as an exercise before, please revisit that again ok. So, this concludes the proof. So, every open set in R has quite a simple structure. It is either the empty set or its going to be a countable union of disjoint open intervals. Many of those open intervals could possibly be of infinite length that means, I am allowing the endpoints to be $-\infty$ or ∞.

This is a course on real analysis, and you have just watched the module on the structure of open sets.