Real Analysis - I
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Lecture - 2.3
Quantifiers

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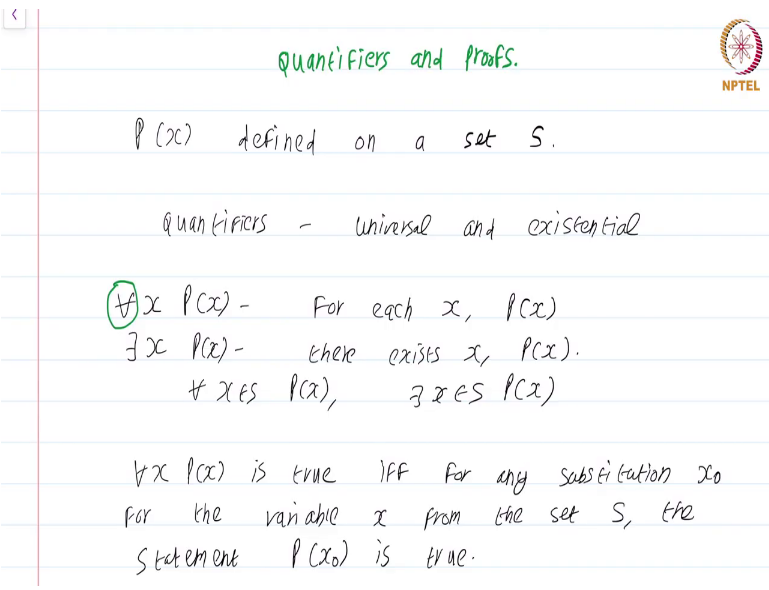

In the last module, we ended by studying properties P(x) defined on a set S. These are sentences that have a variable x and if you substitute an element from the set S this property becomes a statement that is true or false. Always remember, properties are defined over a set S. You can only substitute elements of this set S in the place of x.

Now, there are two other ways to make a property into a statement and these are using what are known as quantifiers. There are two types of quantifiers, there is one universal quantifier and then there is an existential quantifier. These are no doubt familiar to you from your high school mathematics. So, let me just briefly recall what they are.

The universal quantifier is a statement of the type for each x P of x. So, this is to be read as for each x P(x)and there the existential quantifier is written like this, there exists x P of x. Since, the underlying set S is clear, we are ignoring that in the notation. If you are going to be 100 percent precise, you must write for all x in S P of x there exists x in S P of x.

So, let me make a general remark that will help in proofs. This universal quantifier can be read in two ways: for all, for each. Both actually mean the same thing, but I recommend that though you use both, for each reading of the symbol is better because for all gives the impression that you are substituting the entire set S all at once in the place of x.

Whereas for each x gives the more correct impression that you are taking a particular element of x one at a time and then you are substituting in the place of variable . So, this reading for all as for each is somewhat better in my opinion, but the for all usage is so common that it would be really difficult for me to avoid using it unconsciously or you will definitely encounter it in the literature anyway. So, I will use both interchangeably, but I will consciously try to use for each.

What do these quantifiers mean? Well the statement for all x P of x is true if and only if for any substitution x naught for the variable x from the set S, the statement P of x naught is true. So, for all x P of x is true if and only if for any substitution x naught for the variable x from the set S the statement P of x naught is true.

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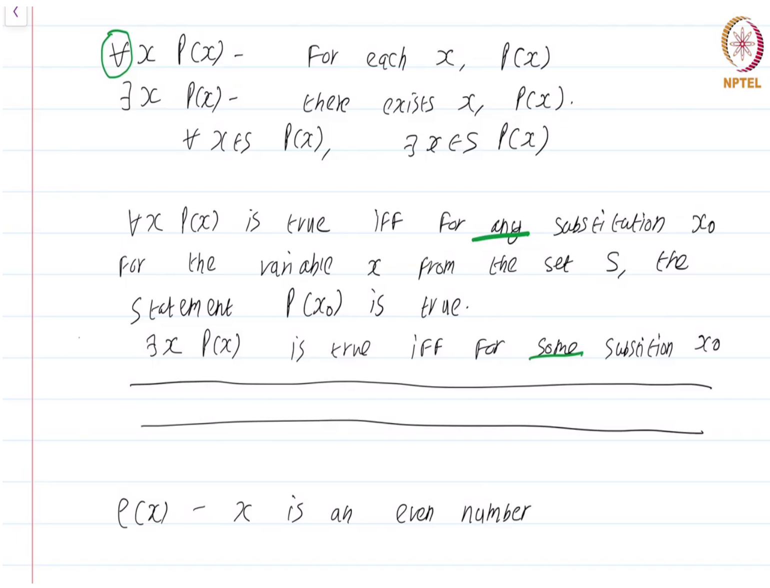

So, what this says is irrespective of what element of the set S you substitute, the statement P(x) is always true. There exists x P of x is entirely similar is true if and only if for some substitution x naught and the rest is same; the statement P of x naught is true.

So, the key difference between the existential and the universal quantifier is that for the universal quantifier, any substitution must yield a value, must yield a statement that is true whereas, the existential quantifier you just need one value.

Now, let us see some examples of existential and universal quantifiers. Let us define what it means for a number to be even using quantifiers. We can define the property e(x) : x is an even number. How do you define this? Well, you have to use quite a complicated statement even for this simple property.

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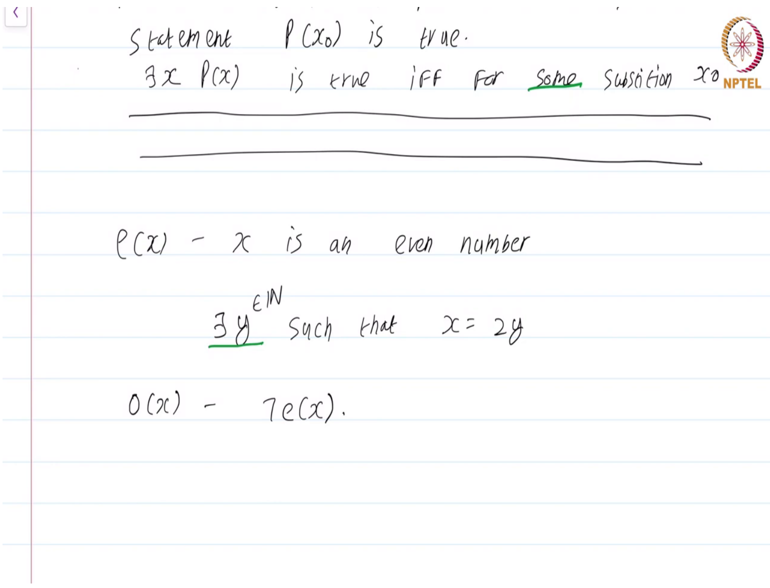

Well e(x) which denotes x is an even number can be defined as there exists y such that x = 2y. I must mention where this y comes from there exist y in N . Now, what does this mean? This means that given an x there must exist a y such that x = 2y.

So, the very definition of the property of being an even number itself involves a statement that involves a quantifier. This is not a problem because if you look back at the definition of a property, what the property demands is merely that for each substitution you get a statement. Here for each substitution, you do get a statement simply because there is a quantifier, existential quantifier already in place. Therefore, what follows will become a statement.

So, we have defined what an even number is, how do you define an odd number we can define a new property O of x to be just negation of e of x. So, we have defined both the property of being an odd number and being an even number and as you can imagine, quantifiers start to build up in rapid fashion when you construct complicated statements. For instance, many complicated statements may involve 3 or even 4 quantifiers.

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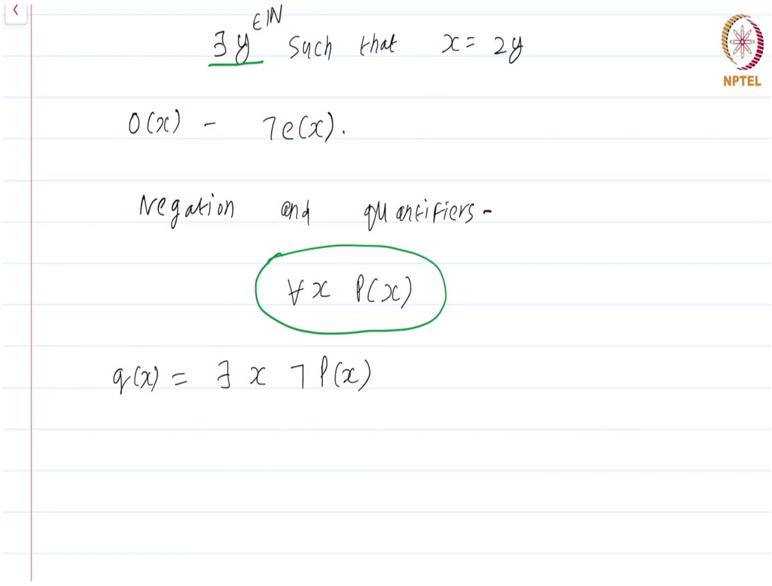

Let us study negation and quantifiers. It is very important that you develop the skill to negate a statement. Remember, negation of a statement is to assert that the statement is false. It has a very easy way to negate statements involving quantifiers that will follow immediately from what the meaning of the quantifiers are.

For instance, suppose you want to negate the statement for all x P of x what do you want to achieve? You want to say that this statement for all x P of x is false. Now, how can a statement that involves a property be false? The only way a statement that involves a property and a universal quantifier can be false if you can find some element in the set S such that the property does not hold for that element. Therefore, you can say there exists x negation P of x. What this is asserting this new statement which we will call q(x), what this is asserting is that there is some element x such that P(x) is false. So, this is precisely asserting that the statement for all x P of x is false. This is how you negate a statement that involves a universal quantifier.

Here is a place where reading for all as for each is beneficial. For each x P of x is true how can that be false? Well there exists x such that P of x is false that is the only way by which that statement can be false.

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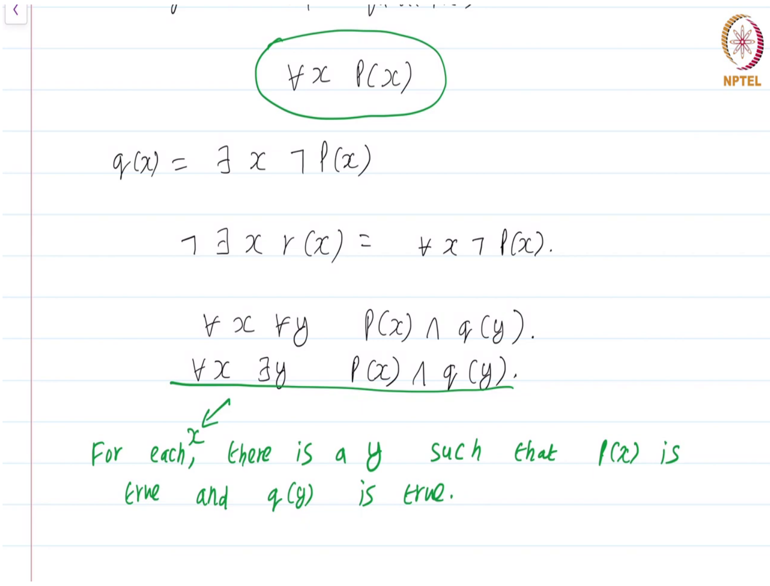

Now, how do you negate a statement that involves an existential quantifier? Suppose I start with the statement there exists x r of x. How do I negate this statement? Well how can it be false that there is no there is an element x such that r of x is true . That will be false only if irrespective of what you substitute r of x is always false. So, the negation of this statement is nothing but, there for all x negation P of x.

So, you see that the universal quantifier and the existential quantifier are rather like evil twin brothers of each other. You can convert one to the other by negation and pushing the negation inside. Remember, you have to push the negation inside also .

What if you have nested quantifiers? Now, here you have to be very careful and understand the meaning of the statement. For instance, let us take a statement that says for all x for all y P of x and q of y what is this statement trying to say? This statement is trying to say that for each x and for each y, the statements P of x and q of y are both true. This is simple enough, but what if I nest a universal quantifier with an existential quantifier. For all x there exists y P of x and q of y. What does the statement mean? Well, let us write it down in English first to understand what the statement means.

This statement means for each x there is y such that P(x) is true and q(y) is true. So, what this says is for each choice of x, there is a y such that P(x) is true and q(y) is true.

So, remember this statement is asserting that given a choice of x, there is a choice of y such that the statement is true. It does not mean that the choice of y is fixed for each choice of x . The statement merely says that if you pick an x, there is a possibility of picking y also such that P of x and q of y is true. It does not say that this choice of y will be unchanging, it will depend on x.

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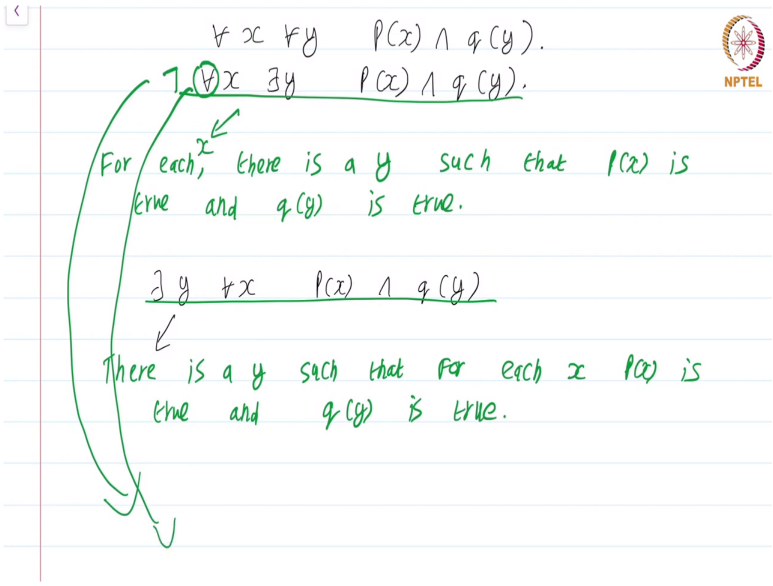

On the other hand, let us say I reverse this statement and say there exists y for all x P of x and q of y. This statement when you translate it into English is saying the following: there is a y such that for each x P(x) is true and q(y) is true. So, this statement says that there is a y such that for each x P of x is true and q of y is true. This is saying that the choice of y is independent of the choice of x. You can find a y such that for whatever choice of x you choose P of x is true and q of y is true.

This statement is far stronger than the previous statement, where all that is required is that for any particular choice of x you can find a y such that P(x) is true and q(y) is true. Consequently, when you negate such complicated statements you have to take into account the order of the quantifiers. The order of the quantifiers are very important.

I will emphasize this when we come to real life situations rather when we come to mathematical situations, I will not artificially introduce a real life situation here and as illustrate with real life statements, but rather I will introduce , what the meaning of these statements mean when we talk about real mathematical definitions and theorems where such statements will crop up I will illustrate with those examples when the time comes.

For the time being, I will just illustrate how you can negate such statements. Suppose you want to negate this statement. What does that mean? That means, you want to assert that the statement for all x there is y P of x and q of y is false. How can that be false? Well remember, how to negate? You change this universal quantifier into an existential quantifier. So, you write there exist x and push the negation inside this is what you do.

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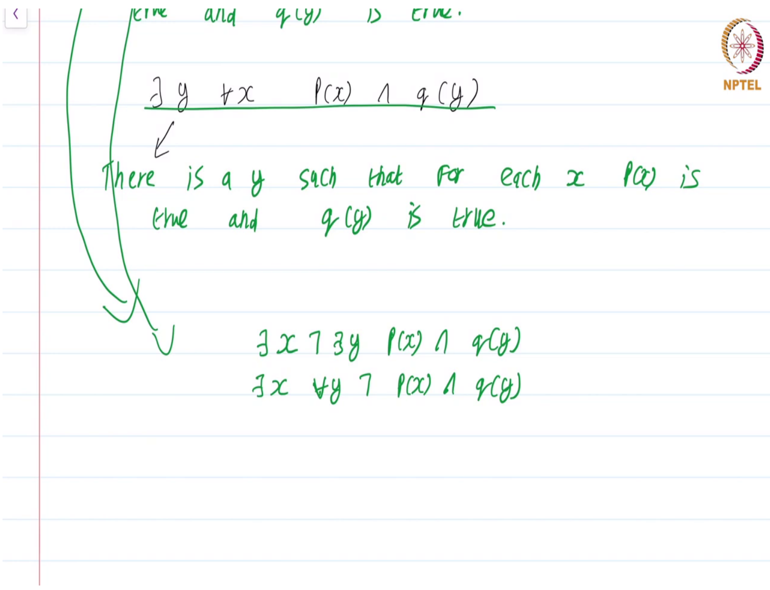

Now, you still have the job of negating there exists y P of x and q of y how do you negate this statement? Well, same procedure you just push the negation inside and change the existential quantifier into the universal quantifier. You just write there exists x for all y it is not true that P of x and q of y.

I strongly recommend that when you are in a situation, you have to negate a complicated mathematical statement, convert it to a notational form like this and do it algorithmically. Once, you have done it sufficiently many times, it will be second nature to you that you will be able to negate statements merely by understanding the meaning of each statement. Till that time comes, I recommend that you do it procedurally and algorithmically.