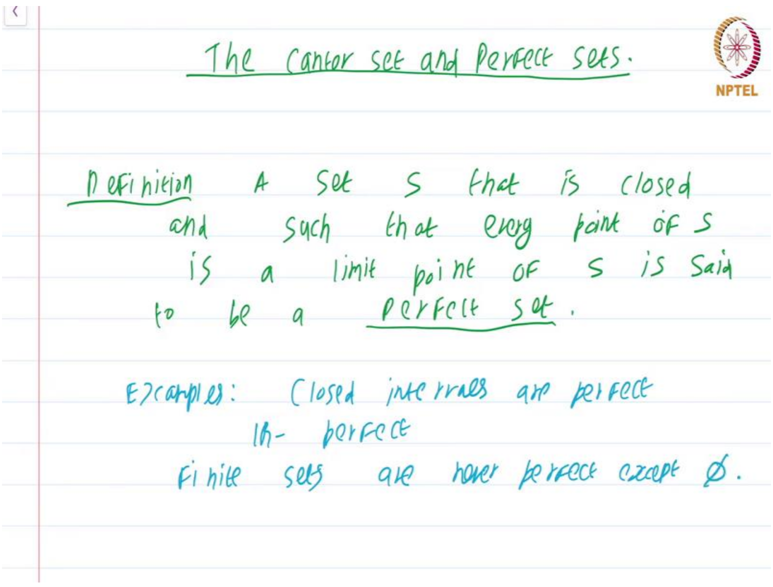
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 20.1
Perfect Sets and the Cantor Set

(Refer Slide Time: 00:14)

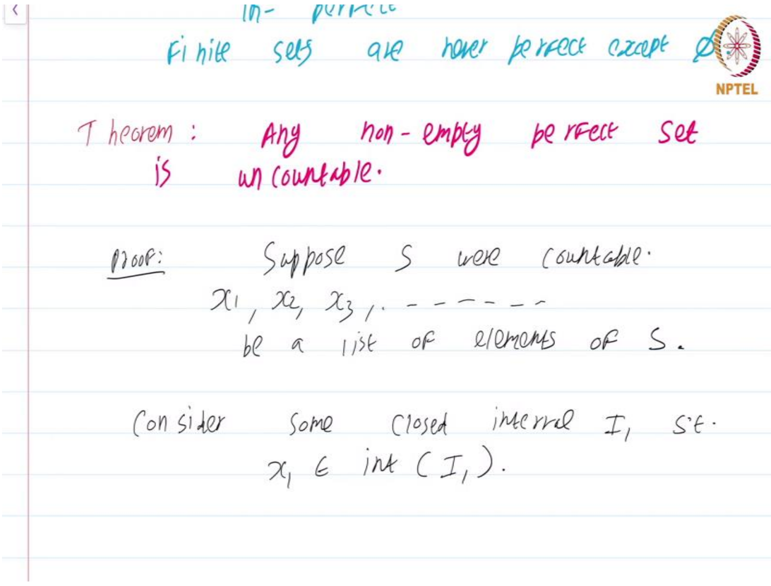


In this module, I am going to introduce yet another type of sets defined using closed and open sets that is another topological property. So, the definition is as follows. Definition : a set S that is closed and such that every point is a limit point of S is said to be a Perfect Set.

Now, I would say that this is not the perfect choice of terminology. Mathematicians, if not anything are never known for creative naming of things. So, I do not know why these are called perfect sets, but this is what we have been handed down by our ancestors ok.

So, examples, well, closed intervals are perfect. The whole of R itself is perfect and for non examples, finite sets are never perfect except of course, the pathological empty set. The empty set is also a finite set ok. Now, we are going to prove that a perfect set has always got to be uncountable.

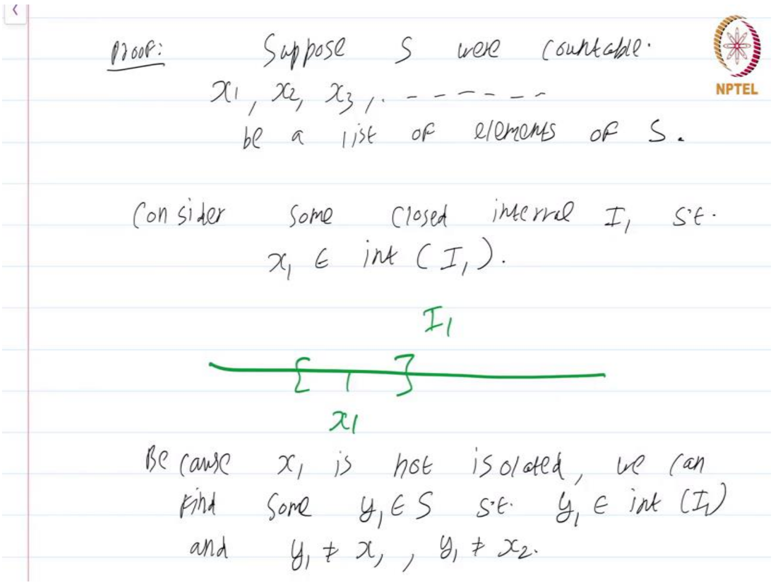
(Refer Slide Time: 02:25)



This is a nice theorem that illustrates a number of things that we have been using. Theorem, any non empty perfect set is uncountable. Let us prove this. Suppose S were countable, then list out the elements $x_1, x_2, x_3, \ldots$ so on list out these elements be a list of elements of S. Now, note that each 1 of these elements $x_1, x_2, x_3, \ldots$ they are all going to be limit points of the set S right.

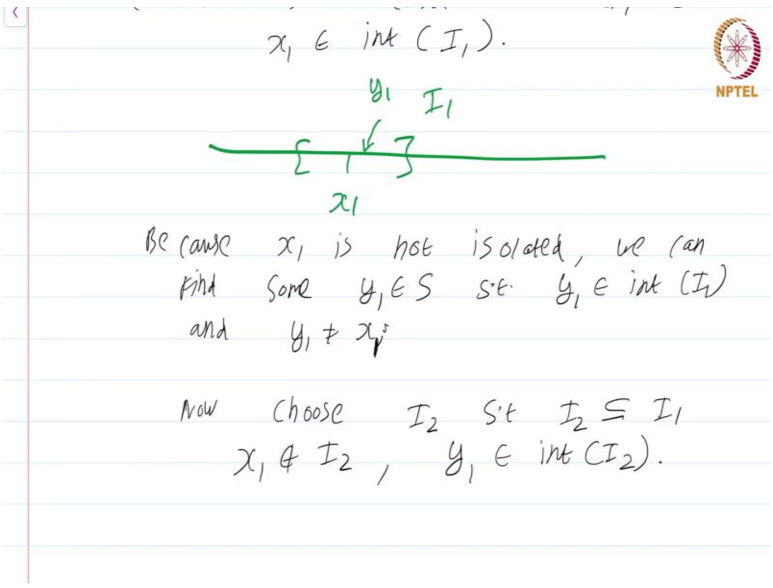
So, what we do is the following consider some closed interval I_1 such that $x_1 \in int(I_1)$. It is not one of the end points of the closed interval I_1, somewhere inside you choose x_1. So again a picture which is worth a thousand proofs.

(Refer Slide Time: 03:57)



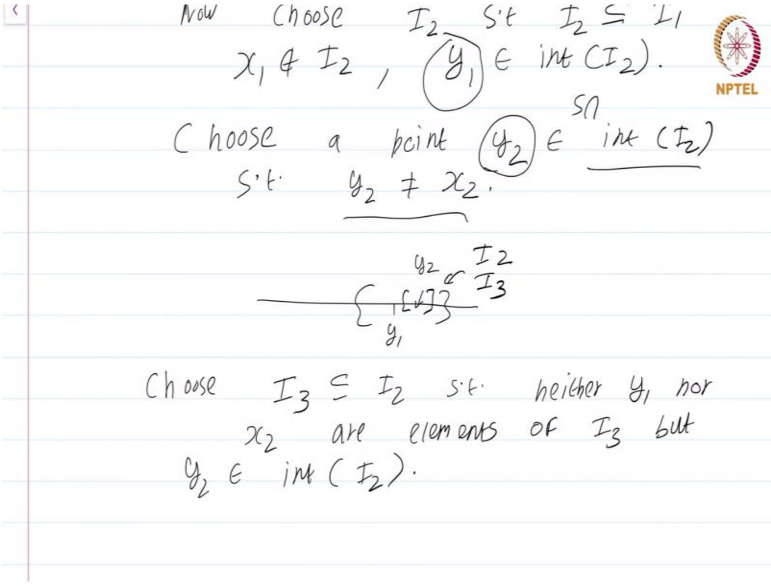
So, you have x_1 and you have this interval I_1 ok. Now, because x_1 is not isolated, it is a limit point, we can find some $y_1 \in S$ such that y_1 is there in interior of I_1 and $y_1 \neq x_1$. Actually this second part is not needed. I will do it in the next step, such that y_1 is not x_1 ok. So, there will be some y_1 simply because this x_1 is not an isolated point.

(Refer Slide Time: 04:48)



I am choosing this y_1 not to be equal to x_1 that is what the meaning of x_1 is the limit point is ok. Now, choose I_2 such that $I_2 \subseteq I_1$. $x_1 \notin I_2$ and y_1 is in the interior of I_2. So, now we have chosen this I_2 such that $I_2 \subseteq I_1$ and $x_1 \notin I_2$ and y_1 is in the interior of I_2.

(Refer Slide Time: 05:49)



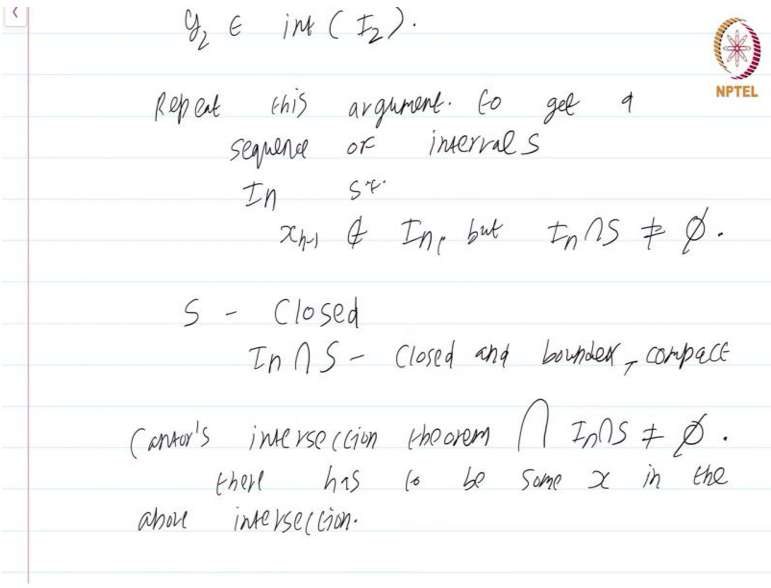
Now, what we do is the following choose a point y_2, which is there in interior of I_2. Such that $y_2 \neq x_2$ choose a point like this. Why does such a point exist, well, such a point exists because, y_1 after all is an element of S. I should mention y_2 is also there in S of course, S intersect interior of I_2.

Because, y_1 is there in S it is going to be a limit point of the set S, because it is a limit point of the set S there are going to be infinitely many points of S which will be there in interior of I_2. Because, there are infinitely many points of S, which are there in interior of I_2. I can always pick this y_2 such that $y_2 \neq x_2$ ok.

So, now zooming the previous picture what has happened is we have this I_2 and in this I_2 we have our element y_1 we are choosing a y_2 here ok. Now, as you can guess we are going to choose I_3, we are going to choose $I_3 \subseteq I_2$ such that such that neither y_1 nor x_2 are elements are elements of I_3.

But y_2 is there in the interior of I_2 ok. Now, we are going to choose another interval I_3 like this. Note this is why I required y_2 not to be equal to x_2, if y_2 happen to be equal to x_2. I cannot do this there is no way to do this ok.

(Refer Slide Time: 08:02)

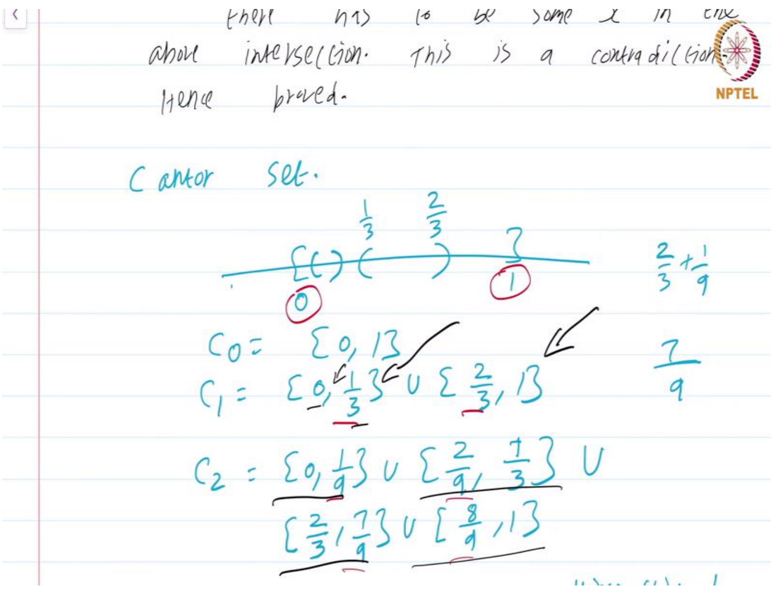


Now, repeat this argument, that means, because y_2 is a limit point of the set S there is a y_3 which is there in this interval I_3 and such that this y_3 is not equal to x_3, ok. And then choose an interval I_4 which is subset of I_3 so on and so forth. So, repeat this argument to get a sequence of intervals I_n, such that $x_{n-1} \notin I_n$, ok. But, $I_n \cap S \neq \phi$. ok.

Now, S, if it were a perfect set, S would be closed, if S were a perfect set it will be closed. Therefore, $I_n \cap S$ will be closed and bounded. And therefore, compact by Heine Borel theorem; that means, by Cantors intersection theorem, intersection of this I_n intersect S is not the empty set, right.

So, there has to be some x in the above intersection, right. But, that is simply not possible why is this not possible? Well, none of the x_n's can be there in this intersection, because x_{n-1}, is not going to be there in I_n. But, this is supposed to be an exhaustion of this list is supposed to be an exhaustion of the elements of S, this simply leads to a contradiction ok, hence proved.

(Refer Slide Time: 10:16)



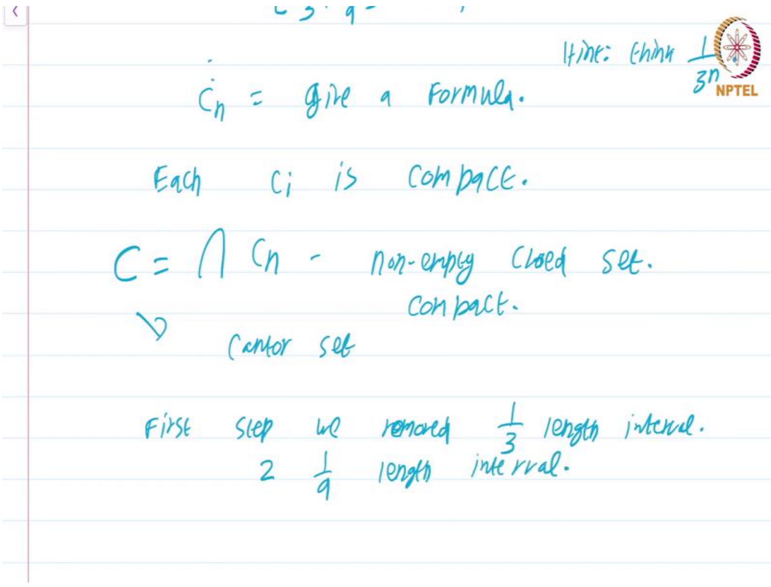
So, this is yet another application of a variant of the nested intervals theorem to show that any perfect set has to be uncountable. Now, I am going to define one of the most fascinating objects in topology called the Cantor set. This will turn out to be an exotic example of a perfect set, it has a number of weird properties one can in fact, spend several weeks exploring all the properties of the Cantor set. Please check the reference below if you are interested in more about the Cantor set ok.

Let us check some properties of the Cantor set, before that I have to define it. Look at the closed interval $[0, 1]$. I just call $C_0 = [0,1]$, ok. Now, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, I just remove the middle one third, I just remove that, that is just $(\frac{1}{3}, \frac{2}{3})$, right.

So, I will be left with $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. I am just opening them removing the middle open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Now, C_2, what I do is the following I remove the middle one third here. The middle one third here, well what is that going to be, well, think about this, that is going to be an interval of size $\frac{1}{9}$ right.

So, it is going to be $\left[0, \frac{1}{9}\right]$ then $\left(\frac{1}{9}, \frac{2}{9}\right)$ has been completely removed from this ok, $\left(\frac{1}{9}, \frac{2}{9}\right)$ has been completely removed from this. So, what will remain is union $\left[\frac{2}{9}, \frac{3}{9}\right]$ which is just $\frac{1}{3}$ right which is just $\frac{1}{3}$. And that is going to be union $\frac{2}{3}$, Now, I need to figure out what is the next one after $\frac{2}{3}$. So, that is just $\frac{2}{3}+\frac{1}{9}$, if you quickly do the arithmetic this is going to be $\frac{7}{9}$ right. So, this is going to be $\frac{7}{9}$ ok. Now, the open interval $(\frac{7}{9}, \frac{8}{9})$ is removed. So, this is going to be union $\left[\frac{8}{9}, 1\right]$ ok.

(Refer Slide Time: 13:09)



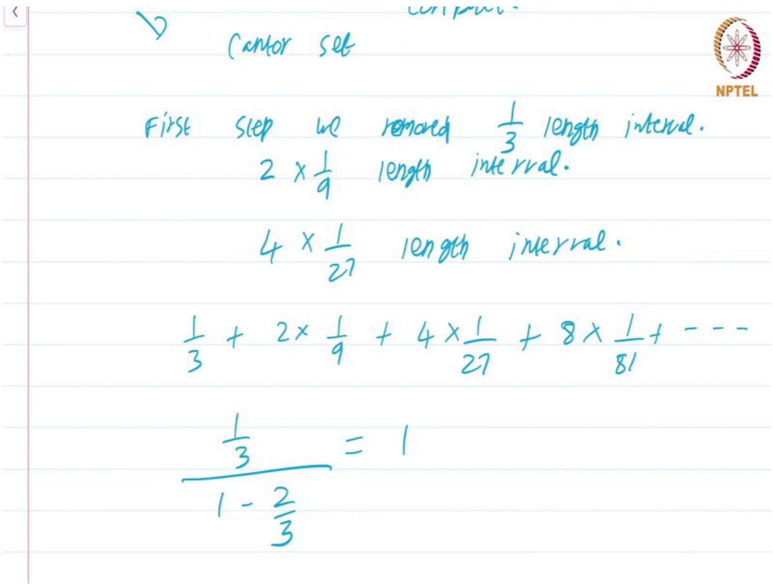
So, I hope the procedure is clear so, you successively define $C_1, C_2 C_3$ so on and in C_n what you do is you look at each piece of the preceding union and remove the middle one third out of it ok. Now, I am going to leave an exercise. Give a formula for this, hint: think $\frac{1}{3^n}$, that is the vague hint I am giving you, give a formula for how C_n is going to look ok. Now, what are the properties of the C_0, C_1, C_2, C_n well each C_i is compact.

Because, it is closed and bounded its closed because, it is a finite union of closed intervals, its compact because its bounded also ok. Therefore, it is closed and bounded therefore, its compact look at intersection of C_n. This is going to be a non empty closed set, why is it going to be a non empty set?

Because, by Cantors intersection theorem again this is going to be a closed set, but it is also a bounded set therefore, its compact this is going to be a compact set it is a closed set because it is an intersection of closed sets. So, we have defined this set C the Cantor set as the intersection of the sets obtained by successively removing the middle one thirds starting with the interval close [0,1] ok.

Now, observe the following, at the first step, we removed one third length interval right. The middle one third the interval length is $\frac{1}{3}$ at the second step. We removed two $\frac{1}{9}$ length interval ok, consequently we ended up with 4 intervals.

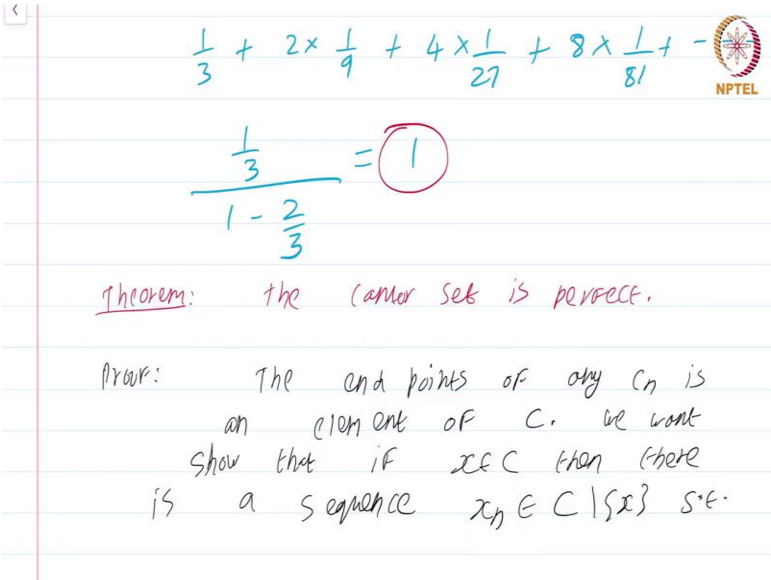
(Refer Slide Time: 15:31)



Then we removed one third of that so, 4 into $\frac{1}{27}$ length interval ok. So, you see where this is going, let us look at the sum of the lengths of intervals we have removed, its $\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + 8 \cdot \frac{1}{81} + \ldots$, ok.

Now, this is just a geometric series the first term is $\frac{1}{3}$ whereas, the common ratio r is clearly $\frac{2}{3}$ so, $\frac{a}{1-r}$ so, its $1 - \frac{2}{3}$ in the denominator and its equal to 1 ok. So, what this means is the lengths of intervals that we have removed from the Cantor set sum up to 1. In some sense we have removed the whole of the interval $[0,1]$ at least from this naive argument, but that is not the interesting thing.

(Refer Slide Time: 16:48)



The more interesting thing I am not saying this is not interesting, sorry about that this is interesting. But even more interesting is the fact that theorem the Cantor set is perfect the Cantor set is perfect, whoa this is fascinating the Cantor set is perfect, which means that its uncountable.

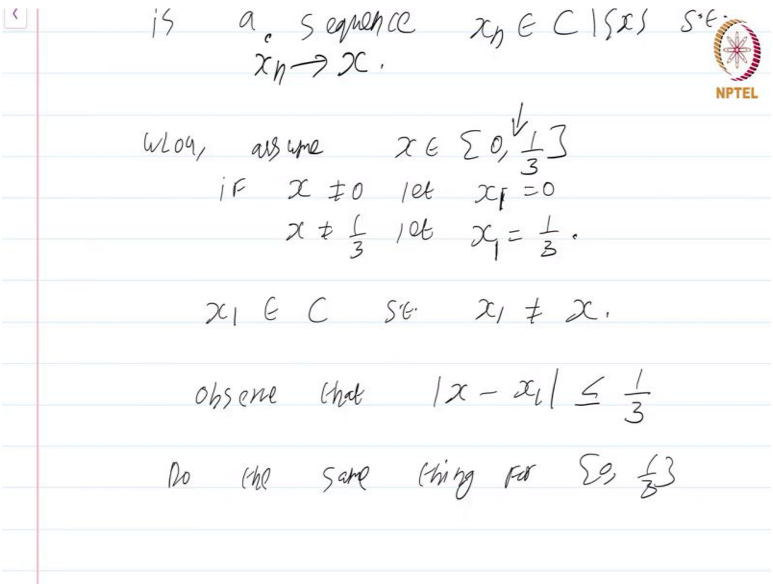
So, even though you seem to have removed the entire length [0, 1] length interval, as the sum of the series shows the Cantor set is nevertheless uncountable. Because, the previous theorem says perfect sets are uncountable ok. But, before that let us come back and think about this Cantor set in some more depth, I said the Cantor set is nonempty by using a very powerful theorem called the Cantor intersection theorem.

But that is pretty stupid of me because, you can show that the Cantor set is nonempty without appealing to any fascinating theorem like this just look at 0 and 1. Just look at 0 and 1 they are definitely going to be there in the Cantor set why are they definitely going to be there in the Cantor set? Because think about it for a moment. The only way by which these endpoints 0 and 1 will not be there in the Cantor set is if they are removed at some point.

But, at each stage we are only removing something from in between the Cantor set, right, not in between the Cantor set, in between the intervals that we have. So, these end point 0, 1 will never be removed because at each stage we are only going to remove something from in the centre, nice, 0 and 1 are there in the Cantor set. But, wait a second does not the same argument hold true for the point $\frac{1}{3}$ and $\frac{2}{3}$ yes it does. At each stage after C_1 , $\frac{1}{3}$ and $\frac{2}{3}$ are never going to be touched, all successive removals will be coming from within the interval.

Therefore, $\frac{1}{3}$ and $\frac{2}{3}$ will remain unscathed and will be there in the Cantor set C as well ok. So, will be $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$ so on and $\frac{8}{9}$. So, first the proof of this, the end points of any C_n is an element in C ok, that is the first point. I want to make in this proof, the end points are all present in the Cantor set. How does this help us prove what we want? We want to show that if $x \in C$, then there is a sequence there is a sequence $x_n \in C - \{x\}$ such that x_n converges to x.

(Refer Slide Time: 19:59)



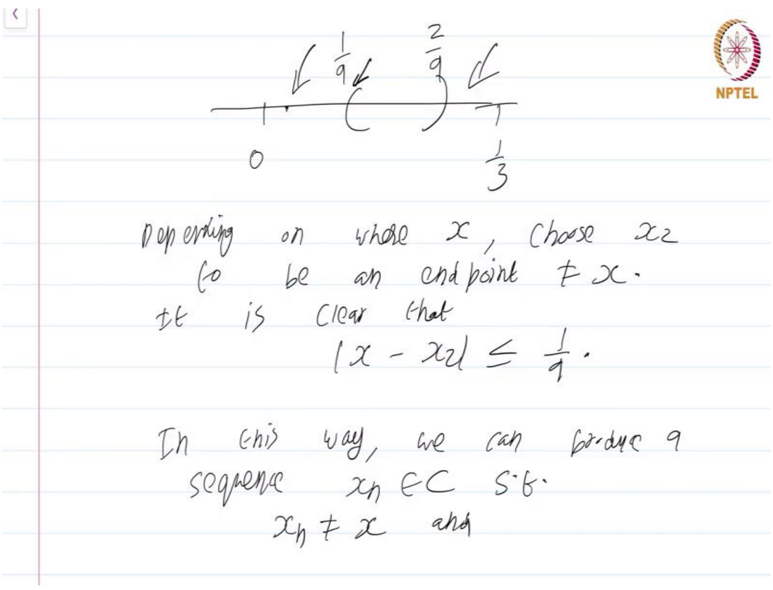
Please recall that this is enough to show that x is a limit point ok. Now, what do we do to produce this sequence x_n, well let us go step by step let us look at this picture again. This point x has to be in C_1 right its it has got to be in C_1, its either there in $\left[0,\frac{1}{3}\right]$ or it is there in $\left[\frac{2}{3},1\right]$ there are only 2 possibilities for this ok.

Now, if it happens to be in $\left[0,\frac{1}{3}\right]$ there are 3 possibilities its either 0 or its $\frac{1}{3}$ or its somewhere in between, those are the 3 possibilities ok. So, so without loss of generality assume that x is an element of $\left[0,\frac{1}{3}\right]$. If $x \neq 0$, let $x_1 = 0$, if $x \neq \frac{1}{3}$, let $x_1 = \frac{1}{3}$, ok. So, if it is one of the end points if choose the other end point ok.

If x is in the middle it really does not matter, the way I have written it x_1 is sort of ambiguously defined, x_1 is sort of ambiguously defined if x_1 is there in the middle. But it does not matter choose any one of the end points if x is actually somewhere in between $\left[0, \frac{1}{3}\right]$, ok. So, what we have done is we have produced a point x_1 in closed interval $\left[0, \frac{1}{3}\right]$. In fact, we have produced a point x_1 in C such that $x_1 \neq x$, right.

Observe that mod $|x - x_1| \leq \frac{1}{3}$. In fact, to be a 100 percent precise, I have to write less than or equal to, it can happen that its one of the end points and the choice of x_1 is therefore, is the other end point excellent. Now, how do you choose x_2 well do the same thing do the same thing for $\left[0, \frac{1}{3}\right]$, right.

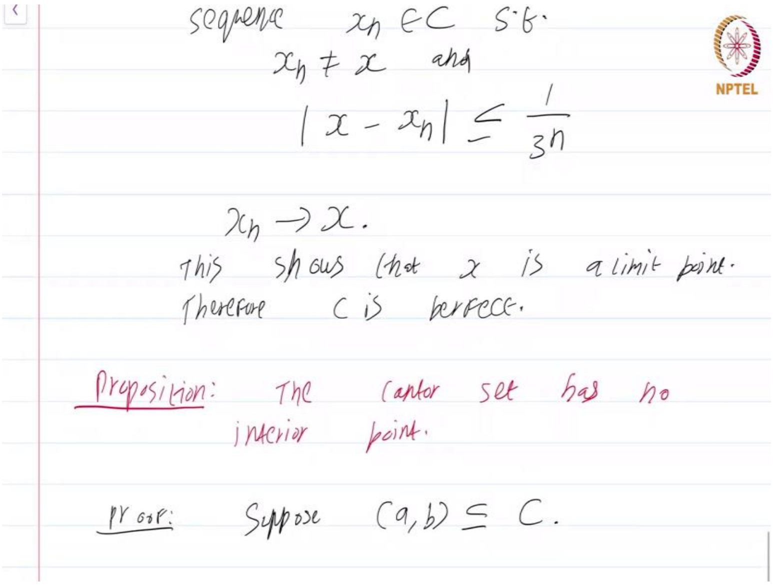
(Refer Slide Time: 22:45)



What do I mean by that look at $\left[0, \frac{1}{3}\right]$, you are removing the middle one third. So, this is going to be $\frac{1}{9}$ and this is $\frac{2}{9}$, right x has got to be either here or here x has got to be either here or here depending on where x is, choose x_2 to be an end point not equal to x ok.

So, if x was actually somewhere here, you could have chosen this $\frac{1}{9}$ as the choice of x_2 ok. It is clear that $|x - x_2| \leq \frac{1}{9}$. In this way we can produce a sequence $x_n \in C$, all are actually, they are actually all end points.

(Refer Slide Time: 24:15)



Sequence $x_n \in C$ such that $x_n \neq x$ and $|x - x_n| \leq \frac{1}{3^n}$, in short x_n converge to x. So, we have found the set $x_n \in C$ such that $x_n \neq x$, and $|x - x_n| \leq \frac{1}{3^n}$ which is just saying that x_n converges to x.

This shows that x is a limit point. Therefore, C is perfect. So, the Cantor set is a perfect set and therefore, it will be uncountable as well. Note that our argument the x_n's we produce are very special they are just coming from the end points ok.

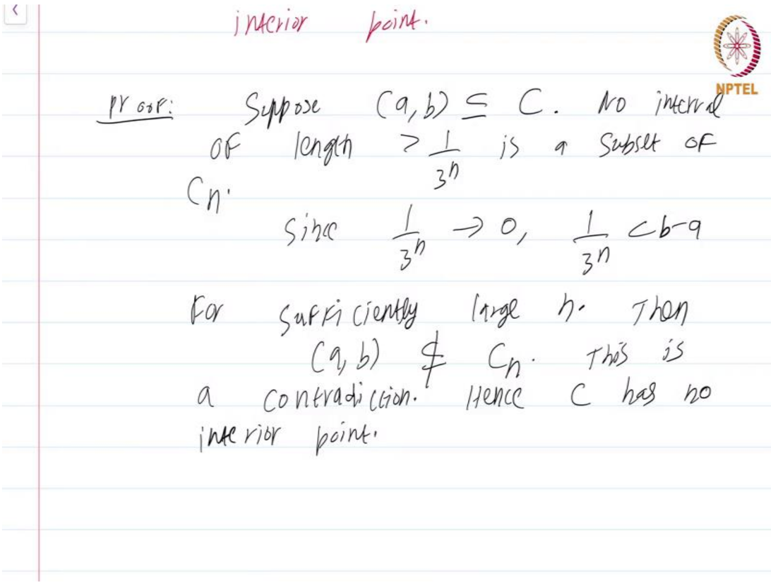
But, you can check once you explore and give a formula for these end points which is an exercise that I have left long ago to you give a formula, it will be very easy to see that these end points, the collection of all these end points is actually a countable set ok.

So, there are points in the Cantor set other than these end points, you might think that there are only these end points, but that is not true there are points other than these end points. Let me just end with one more exotic property of the Cantor set, this is rather easy to see, because the way it is defined itself will give this theorem and I am hesitant to call this a theorem.

Let me just call it proposition the Cantor set has no interior point, well the proof of this is very easy. Proof: suppose it had an interior point. That means, some open interval (a, b) is going to be a subset of the Cantor set. That is the only way by which the Cantor set can have an interior point.

Suppose $(a, b) \subset C$, this is simply not possible. I am not even going to write down the full proof, I am just going to focus here. Notice that at each stage in the construction of the Cantor set, the longest interval will be or not the longest interval each interval will be of length $\frac{1}{3^n}$ ok.

(Refer Slide Time: 27:16)



No interval of length greater than $\frac{1}{3^n}$ can be contained in C_n ok. So, let us write that no interval of length greater than $\frac{1}{3^n}$ is a subset of C_n. So, since $\frac{1}{3^n}$; obviously, converges to 0, $\frac{1}{3^n} < b - a$ for sufficiently large n ok.

Then, (a, b) cannot be a subset of C_n, this is a contradiction. Actually this is not a contradiction yeah. This is a contradiction, I have assumed that I could have rewritten this proof as a direct proof by saying that no interval (a, b) will be a subset of C_n. Therefore, no interval (a, b) will be a subset of C.

But, anyway since I have started with suppose $(a, b) \subset C$. I will end with this is a contradiction. Hence, Cantor set C has no interior point. ok. So, this is just a brief treatment of the Cantor set please check the reference. I have provided for a more extensive treatment. This is a course on Real Analysis and you have just watched the module on the Cantor set and perfect sets.