Real Analysis - I
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Lecture – 19.3
Darboux Continuity and Monotone Functions

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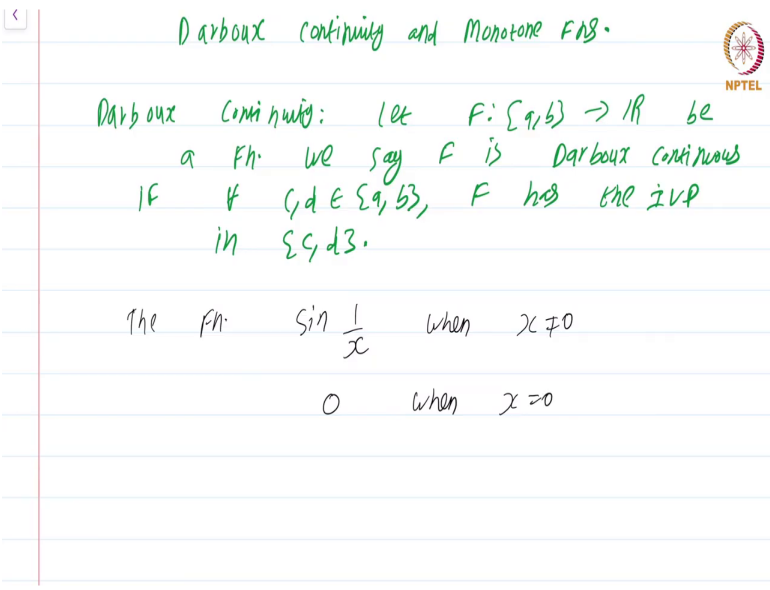

We begin with a definition Darboux Continuity.

Darboux Continuity: Let $F : [a, b] \longrightarrow \mathbb{R}$ be a function. We say F is Darboux continuous, if $\forall c, d \in [a, b]$ F has the intermediate value property (IVP) in [c, d].

That means, if you take two points c and d in [a, b]; look at F(c) and F(d), assume for concreteness that F(c) < F(d), then all values in between F(c) and F(d) will be taken by F as you go along the interval [c, d]. So, we already know that a continuous function is automatically Darboux continuity, we have just proved that. But the converse is not true as the function $\sin \frac{1}{x}$ shows. The function $\sin \frac{1}{x}$ when $x \neq 0$ and 0 when x = 0

is Darboux continuous.

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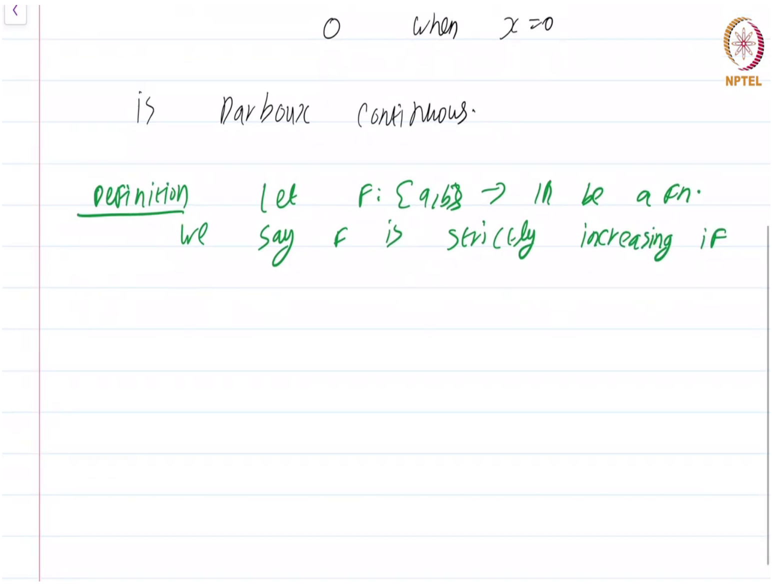

It is Darboux continuous, but it is not continuous at the origin; because it oscillates widely. So, Darboux continuity does not imply continuity. You will later see that derivatives of functions are Darboux continuous; they may not be continuous, but they will certainly be Darboux continuous, they will have intermediate value property.

Now, is there some collection of Darboux continuous functions which are automatically continuous in the sense that, can you show that Darboux continuity plus something else will at least give you continuity? Yes, for that I need a definition.

Definition. Let $F : [a, b] \longrightarrow \mathbb{R}$ be a function. We say F is strictly increasing, if $x < y \implies F(x) < F(y) \; \forall x, y \in [a, b]$.

So, the definition is fairly straight forward, just a remark; similarly we can define strictly decreasing.

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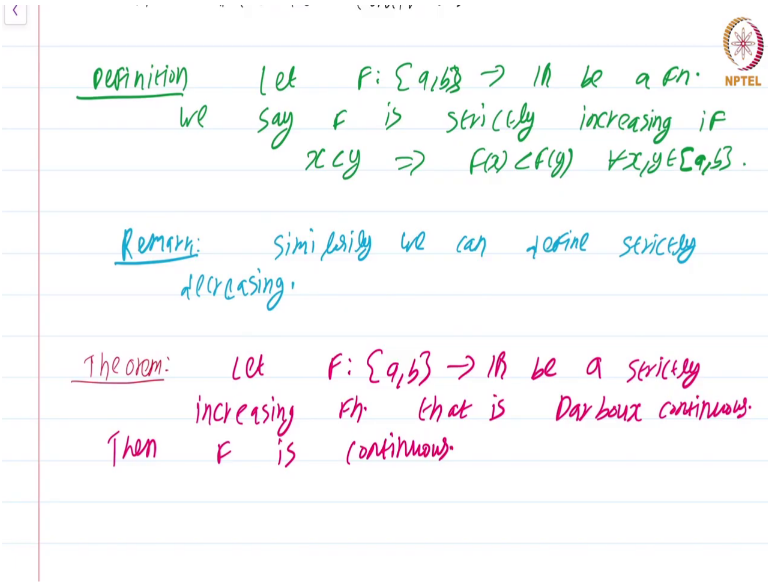

We will see more about these functions when we talk about derivates, especially when we interpret the sign of the second derivative. So, the following theorem is really interesting

Theorem: let $F:[a,b] \longrightarrow \mathbb{R}$ be a strictly increasing function that is Darboux continuous, then F is continuous.

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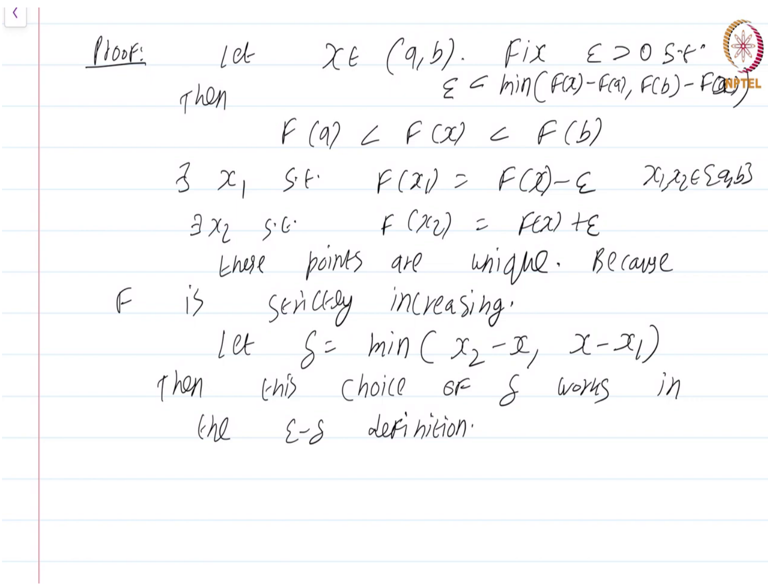

How do we prove this? The proof is not at all hard; we have all the tools that are at our disposal and we will not be using any of them to show this. So, let x be a point in (a,b). First let me take the special case when it is an interior point; the argument for the n points is going to be left as an exercise to you.

So, let $x \in (a,b)$. Fix $\epsilon > 0$. Then observe that F(a) < F(x) < F(b), right. Because F is Darboux continuous; that means $\exists x_1$, such that $F(x_1) = F(x) - \epsilon$ and there $\exists x_2$, such that $F(x_2) = F(x) + \epsilon$, x 1 and x 2 coming from this [a,b],. So, fix $\epsilon > 0$, such that $\epsilon < \min\{F(x) - F(a), F(b) - F(x)\}$.

So, I am just going to choose ϵ so small that, this $F(x) - \epsilon$ and $F(x) + \epsilon$ lie within $[F(a), F(b)]$. If I choose $\epsilon < \min\{F(x) - F(a), F(b) - F(x)\}$ this will be true. Not only does there exist x_1 and x_2 such that this happens, but these points are unique.

Why? Because F is strictly increasing, because of that these points are unique. Now, let $\delta = \min\{x_2 - x, x - x_1\}$. Then this choice of δ, this choice of δ works in the $\epsilon - \delta$ definition.

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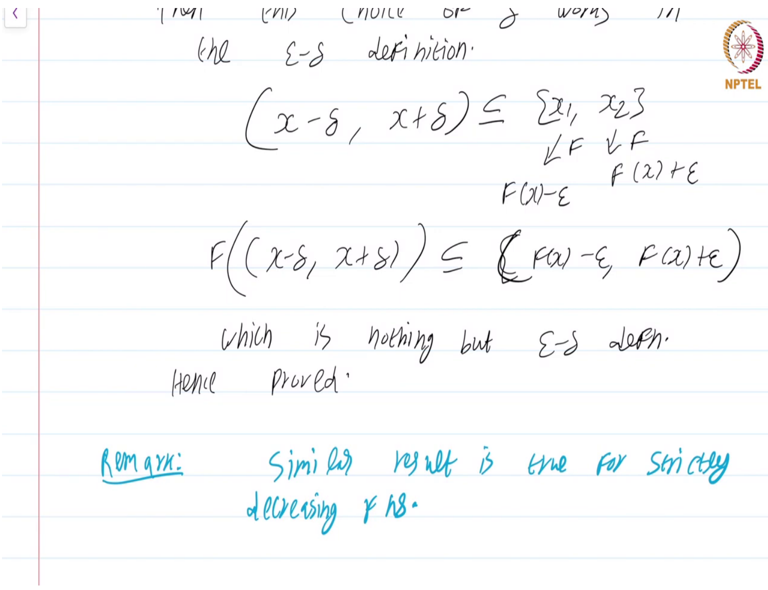

Why does it work? Well, because $(x - \delta, x + \delta)$, if you look at this interval, this is going to be contained in the interval (x_1, x_2).

And at this point it is $F(x) + \epsilon$ and at this point you get $F(x) - \epsilon$, when you apply F, right. By strictly increasing property; that means $F(x - \delta, x + \delta)$, this entire interval has to be contained in $(F(x) - \epsilon, F(x) + \epsilon)$, in fact open, which is nothing but the definition, hence proved.

Remark: similar result is true for strictly decreasing functions. So, I leave it to you to formulate and prove this result, it will be exactly the same thing.

This is a course on Real Analysis and you have just watched the module on Darboux Continuity and Monotone Functions.