# Real Analysis - I

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## Lecture – 19.2
## Intermediate Value Theorem

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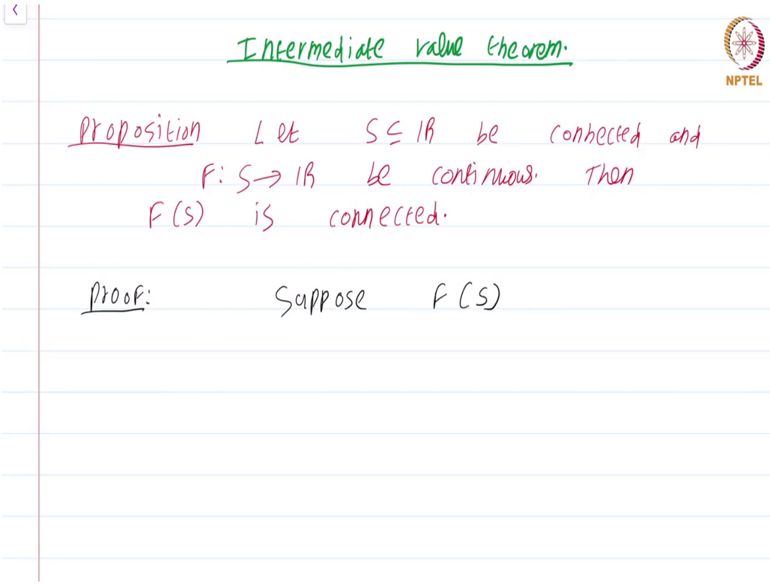


In this module, we are going to prove the famous Intermediate Value Theorem which you might recall was the way we started our course. Before that, let me just state a proposition which is quite easy to prove now that we have extensively studied various equivalent ways of defining connectedness.

Proposition: Let $S \subset \mathbb{R}$ be connected and $F : S \longrightarrow \mathbb{R}$ be continuous. Then F(S) is connected.

The image of a connected set under a continuous mapping is connected. To visualize this you can think of a connected set as a set that is there in one piece. And a continuous mapping cannot rip this into two pieces.

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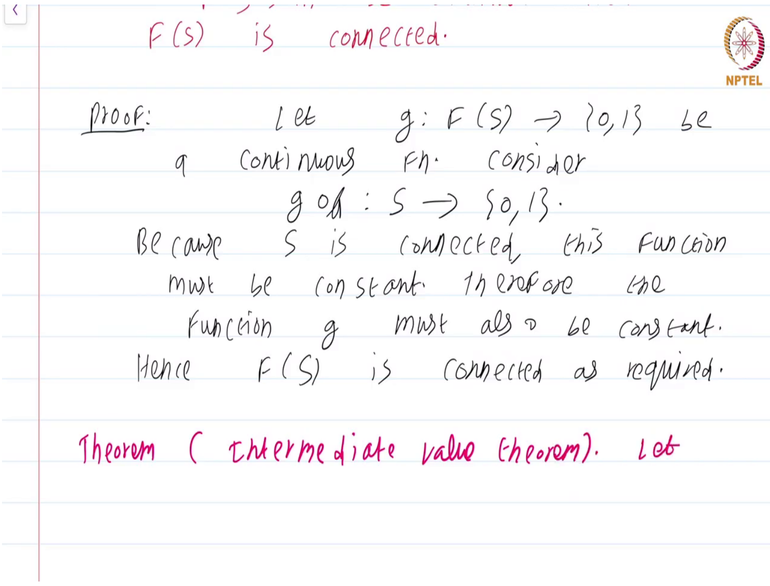


Proof: Suppose F(S) or let me not prove by contradiction; let me give a direct proof. Let $g: F(S) \longrightarrow \{0,1\}$ be a continuous function. Consider $g \circ F : S \longrightarrow \{0,1\}$, because S is connected, this function must be constant. Therefore, the function g must also be constant. Hence, F(S) is connected as required.

Now, how does this help us in our quest to prove the intermediate value theorem? In fact, this proposition makes quick work out of the intermediate value theorem, you can immediately prove the intermediate value theorem.

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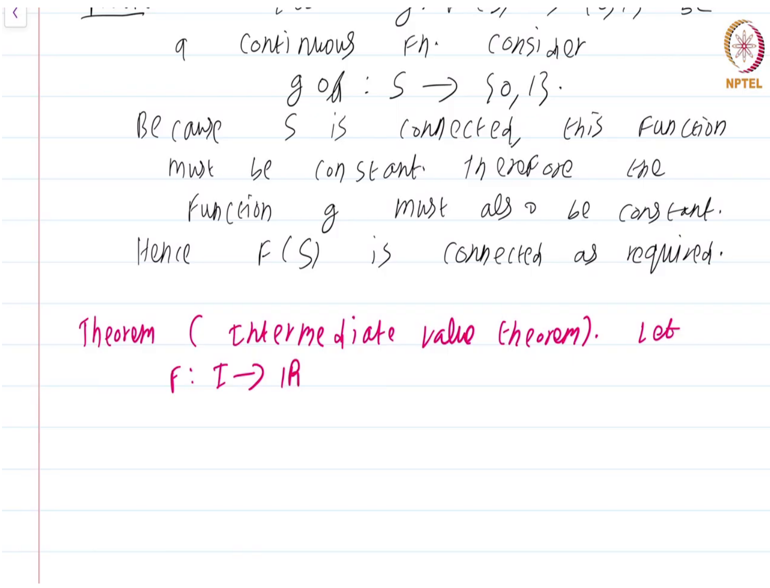

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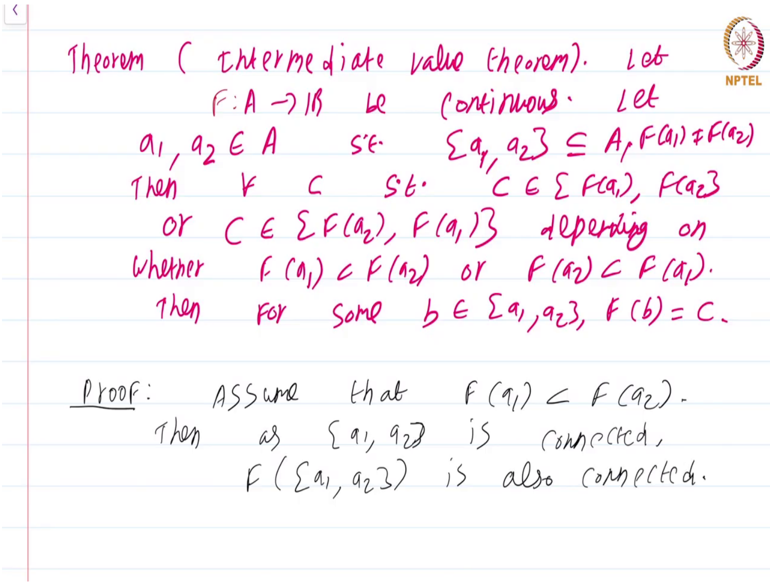


Theorem (Intermediate value theorem): Let $F : A \longrightarrow \mathbb{R}$ be continuous. Let $a_1, a_2 \in A$ such that $[a_1, a_2] \subset A$. Then, $\forall C$ such that $C \in [F(a_1), F(a_2)]$ or $C \in [F(a_2), F(a_1)]$ depending on whether $F(a_1) < F(a_2)$ or $F(a_2) < F(a_1)$.

Depending on whether this you choose C in either the closed interval $[F(a_1), F(a_2)]$ or C in $[F(a_2), F(a_1)]$. If $F(a_2)$ happens to be equal to $F(a_1)$, then this result is entirely trivial, I am not even going to take that case. I am going to assume $F(a_1) \neq F(a_2)$. If $F(a_1) = F(a_2)$, this result is trivial. So, depending on whether $F(a_1) < F(a_2)$ or $F(a_2) < F(a_1)$ choose C in this. Then for some b in closed interval $[a_1, a_2]$, F(b) = C.

Proof: Now just assume that $F(a_1) < F(a_2)$, the other case is perfectly the same. Then as the closed interval $[a_1, a_2]$ is connected, $F([a_1, a_2])$ is also connected; that is just the previous proposition.

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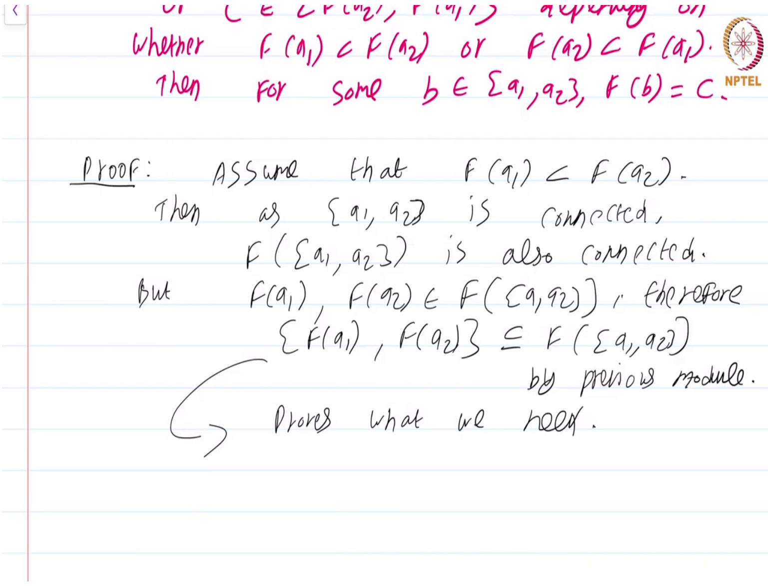


But $F(a_1), F(a_2)$ is an element or both elements of $F([a_1, a_2])$. Therefore, $[F(a_1), F(a_2)] \subset F([a_1, a_2])$. This is just from previous module. Please go back and there is a big theorem in the previous module that immediately gives this ok.

So, this essentially proves what we need  immediately. So, the intermediate value theorem has been reduced to two facts about topology. The first fact is the characterization of connected sets in $\mathbb{R}$; the second fact is that the image of a connected set under a continuous mapping is connected. You can prove this intermediate value theorem without using topology at all and that is directly using the nested intervals property and completeness.

You cannot just use just nested intervals property; you need to use the Archimedean property also that is there in the exercises how to prove it directly. But you will accept that this proof is not only more elegant, but it gets to the heart of the matter, and to precisely see how clarifying this particular proof is please solve that exercise.

This is a course on real analysis. And you have just watched the module on the intermediate value theorem.