Real Analysis - I
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Lecture – 19.1
Connectedness

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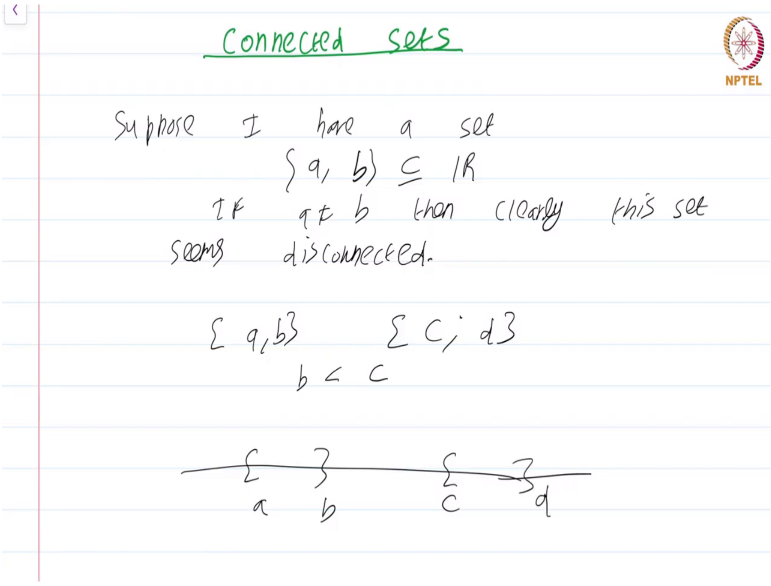

What does it mean for a set to be in one piece? The notion of Connectedness makes this precise. Let us start with an example.

Suppose I have two elements set $\{a, b\} \subset \mathbb{R}$. If a is not equal to b, then clearly this set seems disconnected.

Connectedness is supposed to capture the intuitive idea that the set is in one piece and I am giving you a set which has two different pieces, the piece that contains a and the piece that contains b. Now, thinking further, suppose I have two sets; [a, b] and [c, d] closed intervals such that b is strictly less than c.

So, on the real line I have [a, b] like this and [c, d] like this . So, notice that this set is also in two pieces. This set is also in two pieces and any notion of connectedness should say that this set is disconnected, it is not in one piece.

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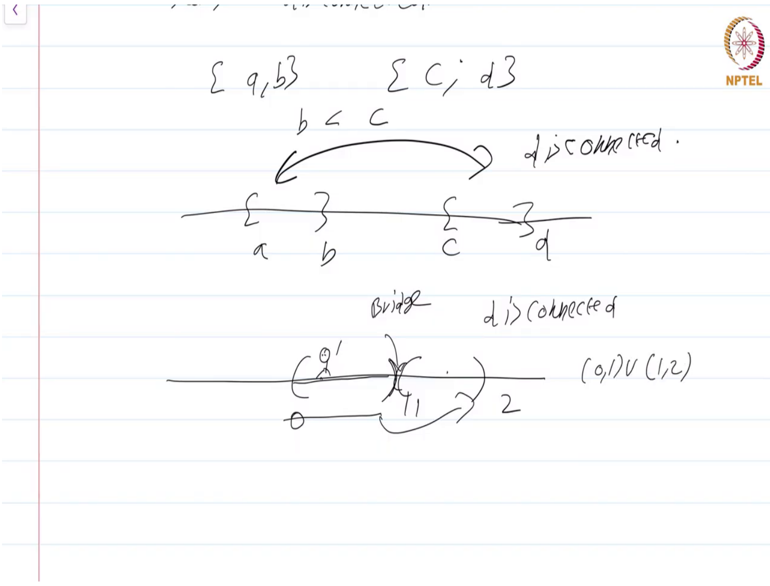

Now, let us see a third example which is much more intuitive. Let us take the set (0, 1) and let me take the set (1, 2), both are open sets. So, these two should be touching each other in fact. Would you say this set is connected or disconnected?

Well, it is actually disconnected. Why would you want to think that it is disconnected? Well, think of it this way. Say you are standing here and you run along this. You run along this set. To go from this piece of the set to this piece you will have to cross this bridge.

You will have to cross this bridge and that particular point that bridge is of actually 0 dimension; it has no width, no height, no nothing, it is just a single point, but nevertheless it is there. You cannot move from this set to this set without crossing over to a point which is not there in this set. So, you want this set $(0,1) \cup (0,2)$ to be also disconnected .

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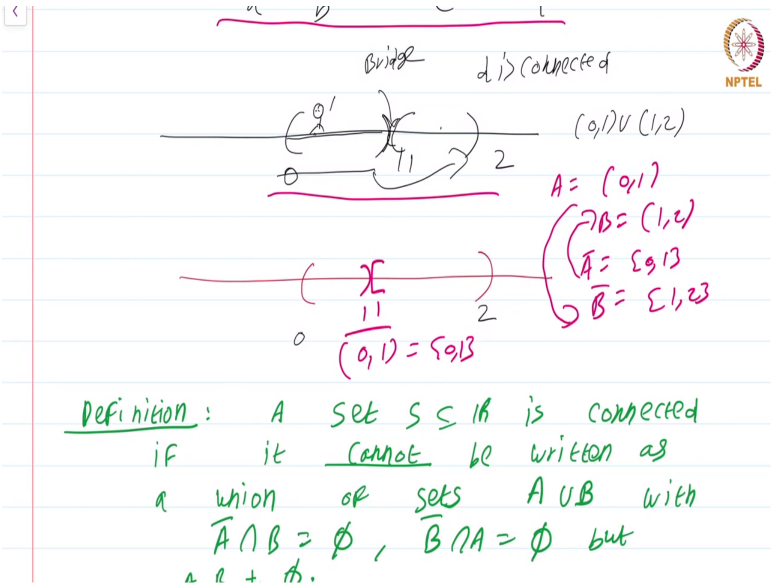

Now, let us see a final example. Let us take (0, 1) again , but this time let us include the bridge. Let us take [1, 2). Now, would you say this set is connected? Well, actually it is. I can just rewrite this set as the set as the set (0, 2) right.

I can rewrite the set as the set (0, 2). Forget that bridge in between. So, therefore, to make precise the notion of connectedness it is a good idea to take into account all these four examples and come up with a definition that captures all of it. So, the following definition does exactly that.

Definition: A set $S \subset \mathbb{R}$ is connected if it cannot be written as a union of sets $A \cup B$ with $\bar{A} \cap B = \phi$ and $\bar{B} \cap A = \phi$, but A, B are both not empty.

So, what is required of a connected set is you are not allowed to write the set as a union of two pieces such that neither of these two pieces sort of touch each other.

And touching each other is captured by saying that $\bar{A} \cap B = \phi$ and $\bar{B} \cap A = \phi$, if both of these happen neither of these two sets touch each other. Now, let us think about what this definition is saying in the examples that we have considered; clearly the set $\{a, b\}$ is disconnected because you can write it as the $\{a\} \cup \{b\}$ and similarly for this, union of two disjoint closed sets .

Now, look at this thing. Obviously, you want to take A to be (0, 1) and B to be (1, 2). $\bar{A} = [0, 1]$ and $\bar{B} = [1, 2]$ and you still observe that $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$ as both empty still.

Now, in the final example where we had taken open (0, 1), but closed I mean [1, 2]; now note that (0, 1) closure will be closed [0, 1] which actually intersects the set B which is having the left endpoint as an element of the set. So, the fourth set would be connected by this definition .

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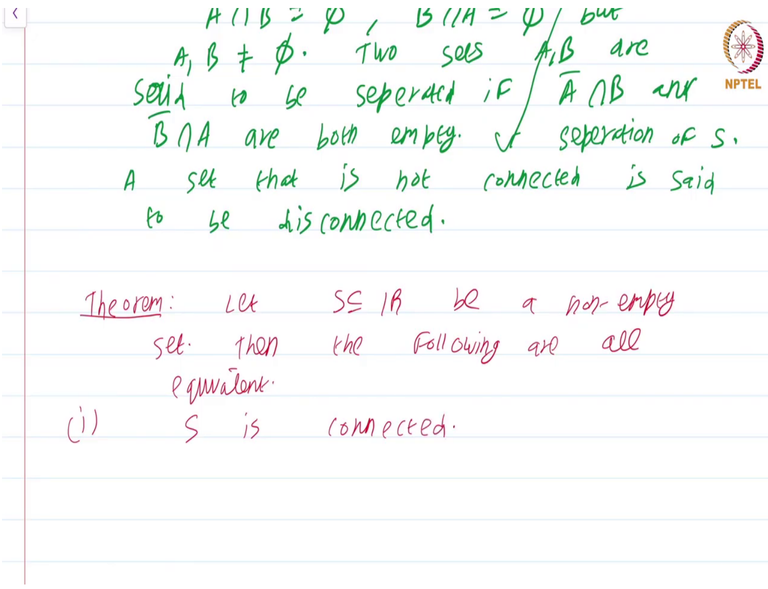

So, elaborating upon this definition, even more two sets A, B are said to be separated if $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$ are both empty. And such a thing taking a set and writing it as $A \cup B$ where A and B are separated set is called a separation.

This is called a separation of S and needless to say a set that is not connected is said to be disconnected. So, connectedness intuitively captures the fact that the set is in one piece.

Now, what I am going to do is I am going to state a theorem that classifies all the connected sets in the real line and the statement of the theorem is a bit long winded because I am going to state several equivalent characterizations.

Theorem: Let $S \subset \mathbb{R}$ be a non empty set, then the following are all equivalent.

(i) S is connected.

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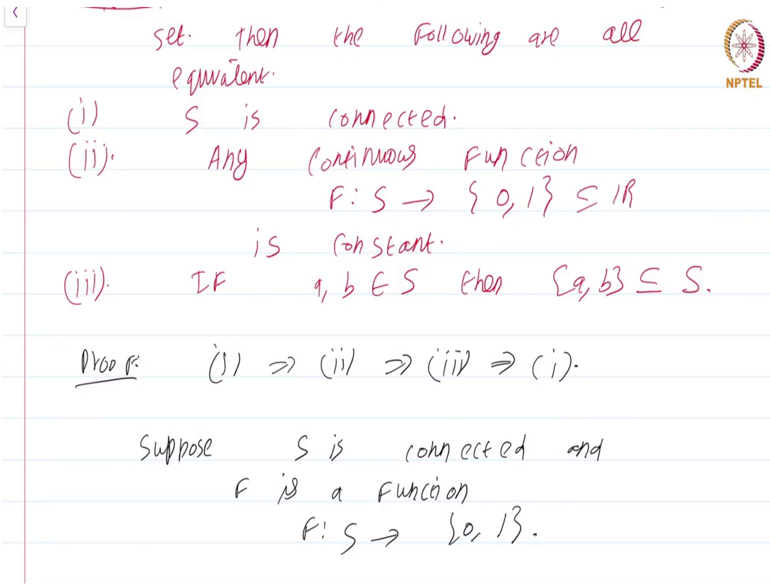

(ii), Any continuous function $F : S \longrightarrow \{0, 1\}$ is constant.

Suppose you have a set that is in one piece and you have a continuous function from that set S whose range or rather the codomain is just the set $\{0, 1\}$.

So, essentially what I am saying is that the codomain is $\mathbb{R}$; however, the range of that function is contained within the set $\{0, 1\}$ is a constant function. So, what it's intuitively trying to capture is that if I have a continuous function on a set that has only one piece there is no way that the value of the function can jump from 0 to 1 as you move across the set.

It is not possible for the function to take two distinct values and only those two values alone. Third condition that will be more geometric. This third condition; if a , b is there in the set S then the closed interval [a, b] is a subset of S. This is entirely believable.

If it were not the case that [a, b] is entirely there in the set S, then what that means is that there is some hole in between and as you walk along the set you are going to fall down in that hole, the set is not in one piece .

Proof; to show that several things are equivalent I have to show that $(i) \implies (ii)$, $(ii) \implies (iii)$, $(iii) \implies (i)$ again. Let us start. So, suppose S is connected and F is a function from S to $\{0, 1\}$.

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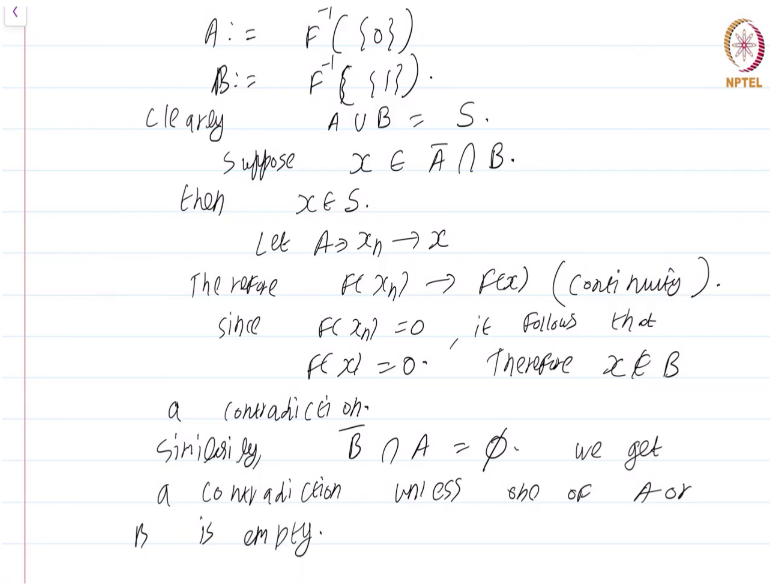

Now, we want to say that the value of this function cannot jump. So, what we do is we define $A := F^{-1}(0)$ and $B := F^{-1}(1)$. We define these two. Now, clearly $A \cup B = S$. Now, suppose $\bar{A} \cap B$ is non empty or why do I want to do that? I do not want to say that what I will do is the following.

Suppose, $x \in \bar{A} \cap B$ then x is certainly an element of S because it is an element of B in particular. Note, why I need to take the intersection with B to assert that it is an element of S is simply because if I take an $x \in \bar{A}$, there is no guarantee that that element x will be in the set S also right because I am taking intersection with B it follows that x is an element of S.

Let x_n converge to x in A. There will be such a sequence because x is coming from the closure. In particular x has to be an adherent point. Therefore, $F(x_n)$ converges to F(x) by continuity. This is just continuity alright. Now, since $F(x_n) = 0$ it follows that F(x) is 0. Therefore, x is not in B.

Therefore x is not in B which is a contradiction, alright. So, what we have shown is that $\bar{A} \cap B$ has to be empty. Similarly, $\bar{B} \cap A$ is empty by the exact same argument. So, we get a contradiction, unless one of A or B is empty right.

The definition of connectedness says that you cannot write it as a union of two sets $A \cup B$, which are both non empty and such that $\bar{A} \cap B$ is empty and $\bar{B} \cap A$ is empty that cannot be done, but since we are writing S as $A \cup B$ with $\bar{A} \cap B$ and $\bar{B} \cap A$ both empty, the only possibility is one of A or B is empty that is this is just saying this is just saying F is constant, a very very fancy way of saying F is constant. So, (i) implies (ii) was fairly easy.

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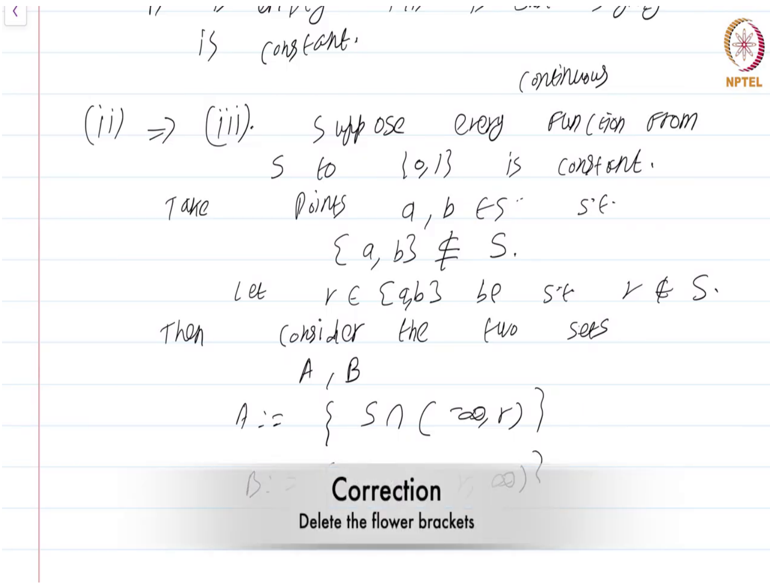

Now, let us do (ii) implies (iii). So, suppose every function from S to $\{0,1\}$ every continuous function of course, crucially I had used continuity here and also let me just for extra clarity write continuous function. Continuity was crucial in the argument.

Suppose every continuous function from S to $\{0,1\}$ is constant. Now, take points a,b in S such that closed interval [a, b] is not a subset of S. So, I am assuming that (iii) is false and I am going to end up with a contradiction. Now, what you do is the following. Let r enclosed in [a, b] be such that $r \notin S$.

Then consider the two sets A,B. $A := S \cap (-\infty, r)$ and $B := S \cap (r, \infty)$. Both A,B are non empty.

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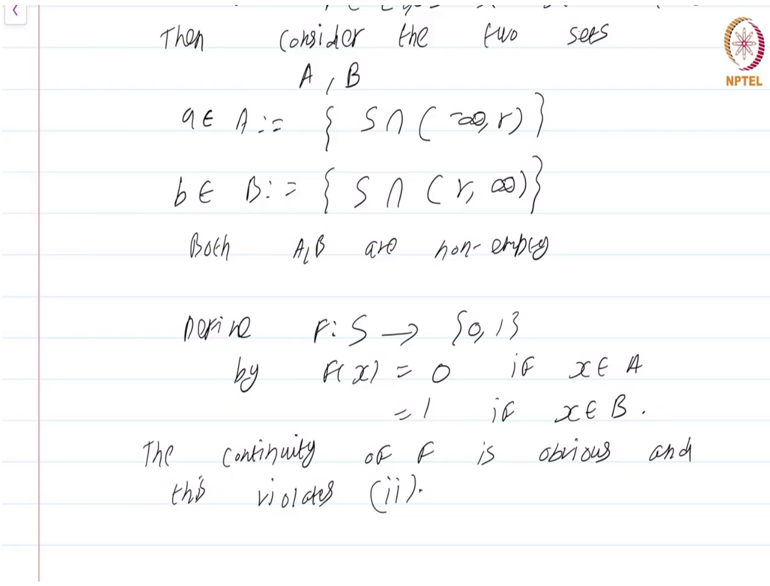

Why is that because $a \in A$ and $b \in B$, that is the way this set A and B have been constructed. Define $F : S \longrightarrow \{0, 1\}$ by F(x) = 0 if $x \in A$ and = 1 if $x \in B$, right. The continuity of F is obvious. It is just constant on two pieces. The continuity of F is obvious and this violates (ii).

So, if every function from the set S to $\{0, 1\}$ is constant then it is impossible for two points a and b to be there in S, but some point in between a and b to be absent. So, this proof was also fairly easy. Now, (iii) implies (i).

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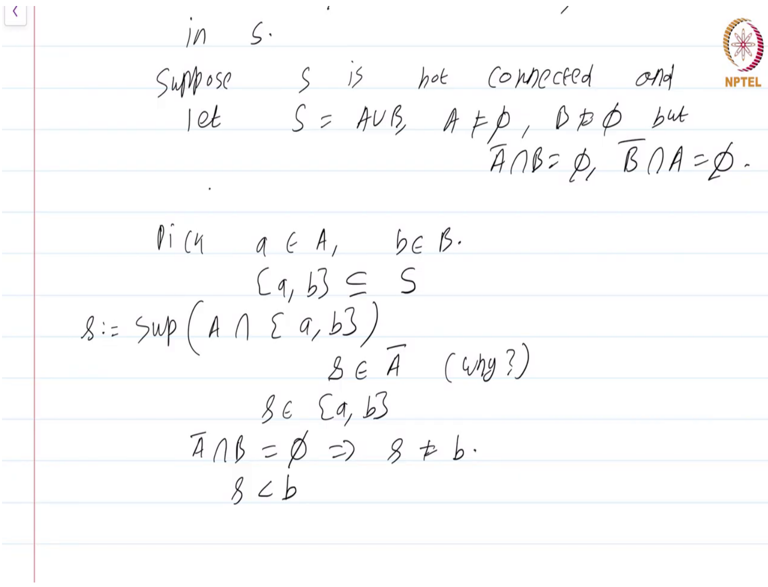

So, suppose we have that if a, b in S, then closed interval [a, b] is also in S . I have to show that the set is connected.

Suppose, S is not connected and let $S = A \cup B$ with A non empty, B not empty, but $\bar{A} \cap B = \phi$ and $\bar{B} \cap A = \phi$.

Suppose we are in this situation. Now, what you do is pick points $a \in A$ and $b \in B$, this can be done because both the sets A and B are non empty. The closed interval [a, b] is definitely a subset of S right simply because that is our hypothesis (iii) right. Now, what we do is we look at $A \cap (a, b)$ and take the supremum of that. Take $\sup(A \cap (a, b))$; call it s. Now, observe the following. This s is definitely an element of $\bar{A}$ right.

I want you to think about why this is the case. It is fairly easy; $s \in \bar{A}$, not only that s is of course, going to be an element of closed [a, b]. . Now, note that $\bar{A} \cap B$ is empty. Why? That is the hypothesis. Therefore, this s is not equal to b. So, s is strictly less than b.

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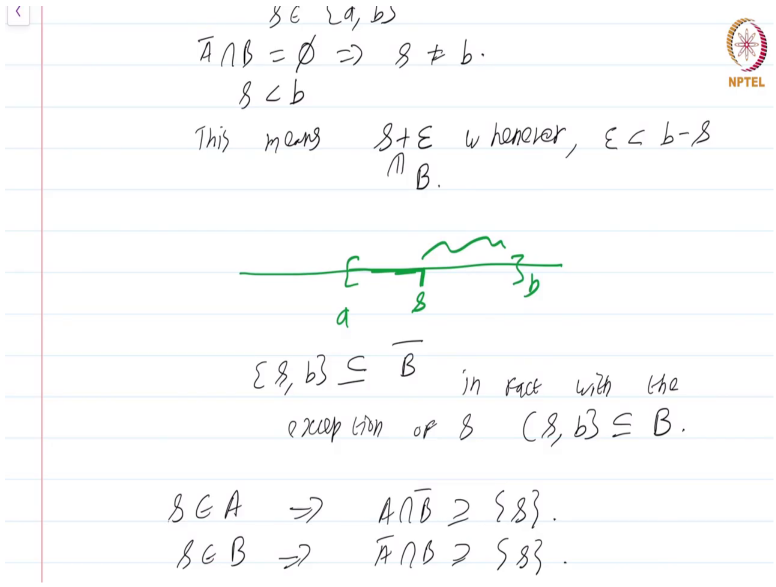

That means, $s + \epsilon$ whenever $\epsilon < b - s$, this has got to be an element of B, right. So, a picture will illustrate what is going on better than me trying to convince you using words.

We have this, [a, b]. We have some portion of this thing which is supposed to be A. I am looking at the extreme most point of the set A that is going to be the supremum of $A \cap (a, b)$ and that I am going to call it S. Now, what I am saying is if you choose ϵ to be less than this then $s + \epsilon$ has to be in B, simply because the set $S = A \cup B$ and this closed interval [a,b] is subset of S right fine.

So, what is this show? This shows that this closed set [s , b] is going to be a subset of $\bar{B}$ right, it is going to be a subset of $\bar{B}$. In fact, with the exception of s; so that means, this open (s, b) this has to be contained in B; no choice because that it cannot be in A.

Now, what does this tell us? There are only two possibilities. Suppose, $s \in A$, well this implies $\bar{A} \cap B$ or rather let me take $A \cap \bar{B}$ will definitely contain the element s. On the other hand if $s \in B$, then $\bar{A} \cap B$ will definitely contain the element s.

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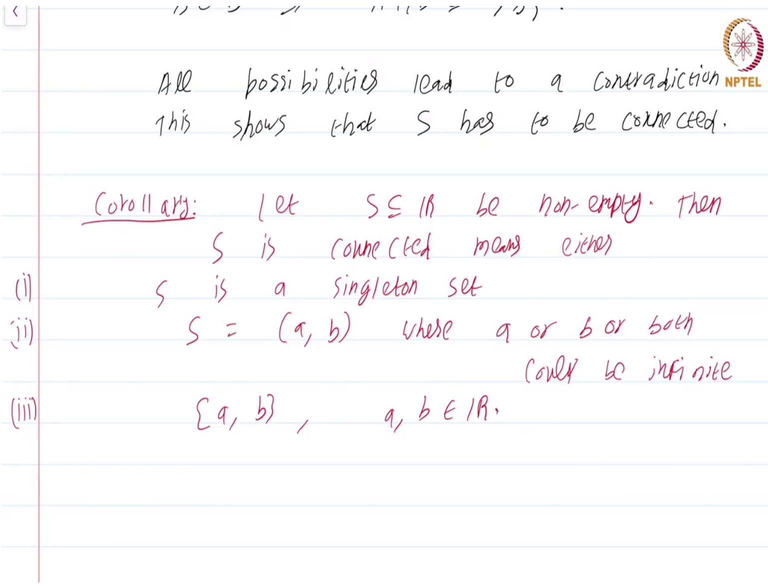

So, which means all possibilities lead to a contradiction. This shows that S has to be connected, excellent. So, the proof was a bit long because we wanted to characterize connectedness in several ways, but nothing in this proof was really difficult.

Let me now state a corollary that will characterize all the sets in R that are connected.

Corollary; this corollary is an immediate consequence of the previous theorem and I am not going to write down the details. Please do that for yourself.

Let S subset of R be non empty. Then S is connected means either

1. S is a singleton set.

2. S is (a, b) an open set where a or b or both could be could be infinite. That means, a could be $-\infty$, b could be $+\infty$, a could be $-\infty$, b could be 0, a could be some fixed number, b could be $+\infty$. So, you got what I am trying to say.

3. it is a closed interval [a, b], $a, b \in \mathbb{R}$. So, the only connected sets are open intervals, closed intervals and singleton sets.

Of course, taking a to be $-\infty$ and b to be $+\infty$ in 2, shows that the whole of $\mathbb{R}$ itself is connected.

So, this corollary is very straightforward to prove from the previous theorem. So, in the next module we shall see the application of this concept of connectedness to prove the intermediate value theorem.

This is a course on real analysis and you have just watched the module on connected sets.