Real Analysis - I
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Lecture – 18.2
Uniform Continuity

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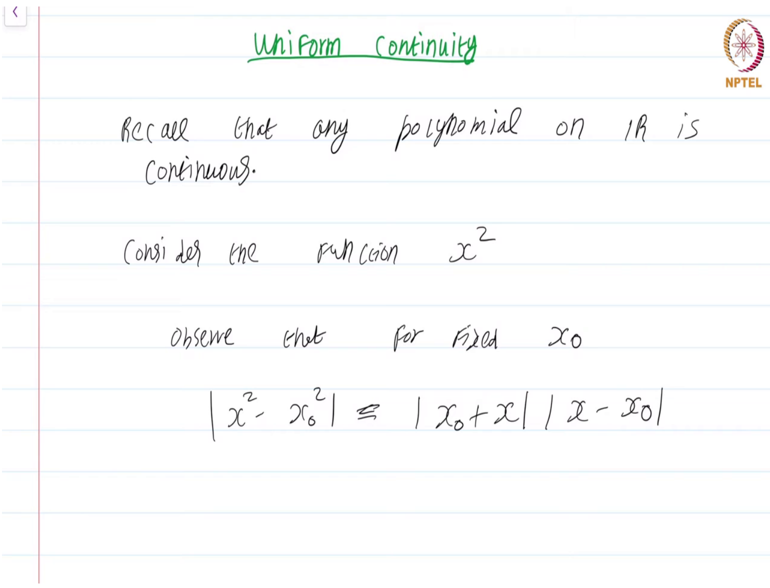

Recall that, any polynomial on $\mathbb{R}$ is continuous. We proved this by first proving that x is continuous and then we applied induction and the algebraic theorems for limits and continuity to show that sums of continuous functions are continuous, products of continuous functions are continuous, and then applied induction to show that any polynomial is a continuous function.

Let us try to prove this directly for one candidate polynomial which we have already done once, let us go through that proof carefully again. Consider the function x^2. Suppose you want to show that this is continuous. Well observe that for fixed x_0,

$$|x^2 - x_0^2| = |x_0 + x||x - x_0|.$$

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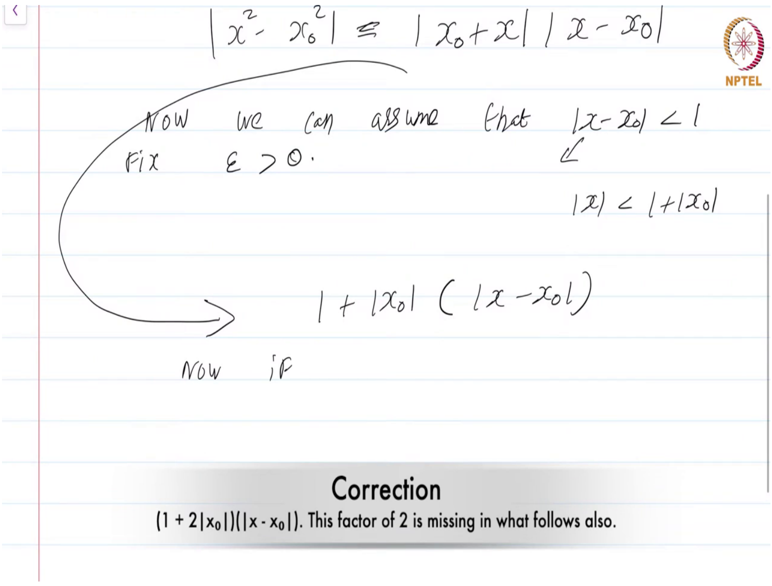

Now, we can assume that $|x - x_0| < 1$. Why can we make this assumption? Ultimately we are going to fix ϵ and find a δ; may as well choose δ to be some quantity that is less than 1 also.

Now, fix $\epsilon > 0$. Well, from the fact that $|x - x_0| < 1$; we immediately get that $|x| < 1 + |x_0|$, right. So, this whole thing is going to be less than $1 + |x_0|(|x - x_0|)$.

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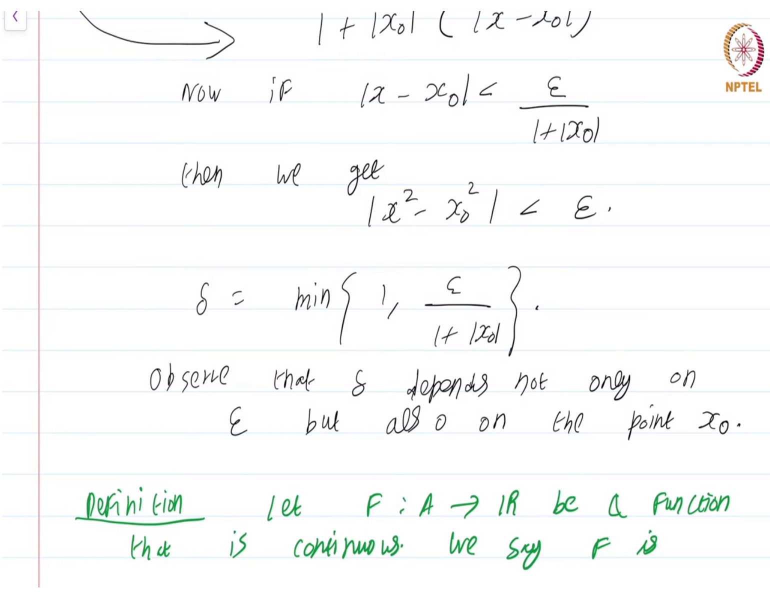

Now, if $|x - x_0| < \dfrac{\epsilon}{1 + |x_0|}$ then we get $|x^2 - x_0^2| < \epsilon$, right. So, our choice of δ corresponding to ϵ is $\min\{1, \dfrac{\epsilon}{1 + |x_0|}\}$, right.

We have found a δ in terms of ϵ. In other words, not only have I shown continuity; I have given you a recipe that for any point x_0, you give me the epsilon, I will return for your δ value which you can compute by a very simple expression that will satisfy the $\epsilon - \delta$ definition of continuity, right.

Observe that, δ depends not only on ϵ; but also on the point x_0. Note, I am not saying that it is impossible to find a δ that works in the definition of $\epsilon - \delta$, which is independent of x_0; that is not what I am saying. It might be possible, we have to show that such a choice is impossible that we will do later on.

But as things stand, our current algorithm or recipe for choosing δ is very much dependent on the choice of x_0. So, this function x^2 is continuous, but its behavior at various points with respect to the $\epsilon - \delta$ definition will change depending on the point.

If x_0 is very very large, then our algorithm is going to return a δ which is much smaller than ϵ as you can see. If x_0 is 10000, $\delta = \dfrac{\epsilon}{10001}$. So, the function x^2 as you start going further away from the origin towards infinity, sort of increases rapidly, that is what this is capturing.

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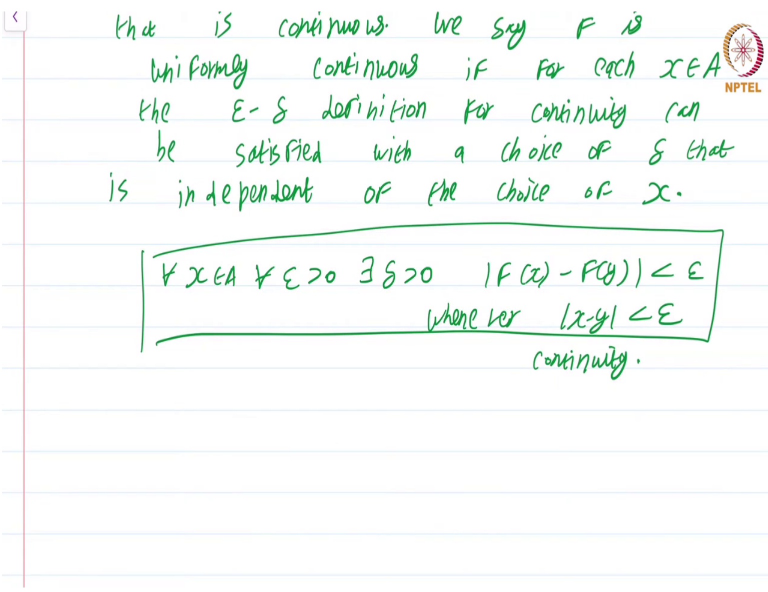

Now, this discussion motivates the following definition.

Definition: let $F : A \longrightarrow \mathbb{R}$ be a function that is continuous. We say F is uniformly continuous, if for each $x \in A$ there exists the $\epsilon - \delta$ definition for continuity can be satisfied with the choice of δ that is independent of the choice of x.

So, first of all at each point of x, the definition of $\epsilon - \delta$ should be satisfied; but the choice of δ does not depend on the choice of point, it is just a function of ϵ. So, let us write this; this definition was given in English, it might be prone to misinterpretation. So, let us write it in logical notation to fully understand without any imprecision what is going on.

Well, what does the $\epsilon - \delta$ definition say $\forall x \in A,\ \forall \epsilon > 0,\ \exists \delta > 0,\ |F(x) - F(y)| < \epsilon \ whenever\ |x - y| < \delta$. This is the usual definition of continuity.

What it says is, for each point x in A and for each choice of ϵ greater than 0; there is a δ greater than 0, such that something happens. Now, that this is the definition of continuity, a slight twist would give you the definition of uniform continuity. The choice of delta should be independent of x.

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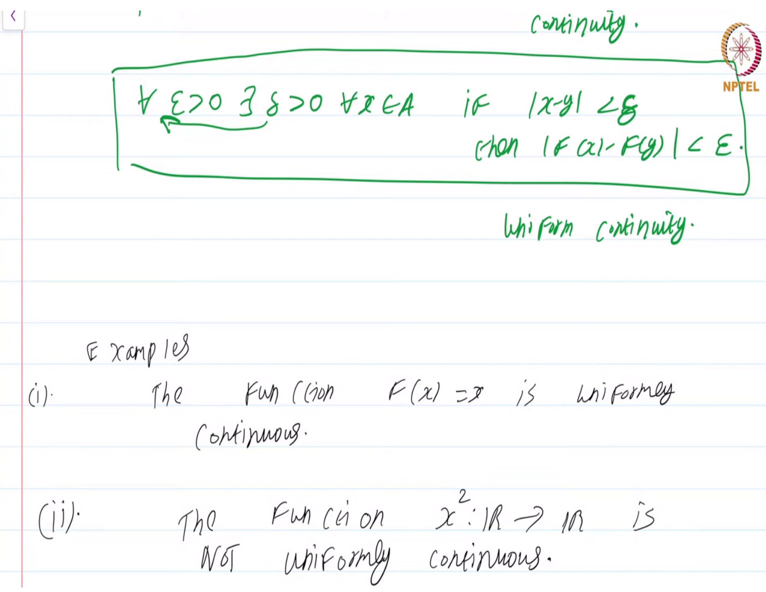

So, what you do is; for each $\epsilon > 0$, there exists $\delta > 0$ for all x in A. Note what has happened, you fix an ϵ, there is a corresponding $\delta > 0$, such that for all $x \in A$, if

$|x-y|<\epsilon$; then if $|F(x)-F(y)|$ is less than δ, here also I made the same mistake delta here is less than epsilon right.

So, this captures uniform continuity. What this says is that, the choice of δ depends only on the choice of ϵ and the choice of point is not into the picture at all. Now, are there uniform continuous functions at all? Well examples

1. The function F(x) = x is uniformly continuous. Well, this is very obvious; because here you can choose δ to be just ϵ right, you do not need to do anything complicated or sophisticated here.
2. The function $x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is not uniformly continuous. Well, is this clear? We just saw this right; no as I remarked there, we have not asserted that it is impossible to find a delta that is independent of the choice of point. It just so happens that our recipe for doing that, depends crucially on the choice of point x_0. Now, we have to make or rather prove that it is impossible to choose δ that is independent of the choice of point x_0.

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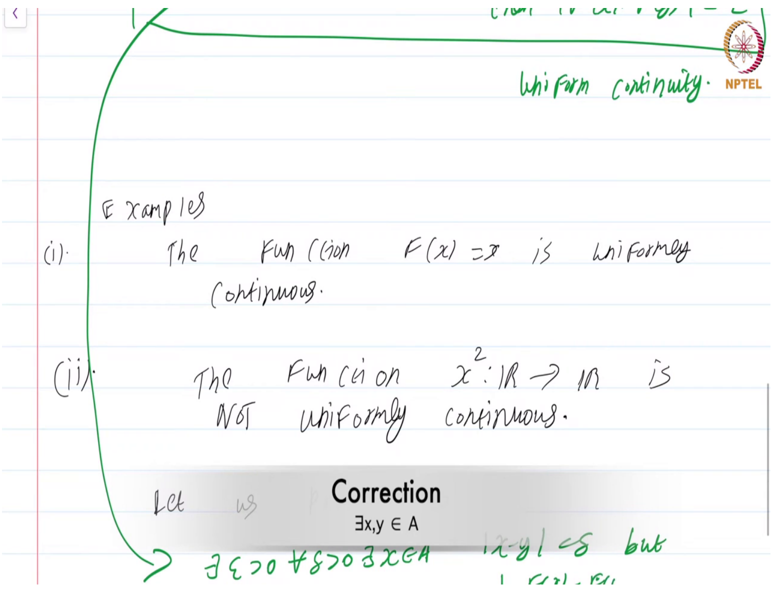

Let us prove this. Now, how can a function fail to be uniformly continuous? Let us try to understand this, before we proceed to prove that $x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is not uniformly continuous. For that let us look at the logical version of uniform continuity.

For each $\epsilon > 0$, there exists $\delta > 0$ for all x in A, such that something happens. What is the negation of this? Well we are now experts at negating. Let me negate it here, $\exists\epsilon > 0$, such that $\forall\delta > 0 \;\; \exists x \in A$, such that $|x - y| < \delta$, but $|F(x) - F(y)| \geq \epsilon$, right.

What is this saying, the only way by which the definition of uniform continuity can fail is; if there is some epsilon that plays spoilsport. So, there exists an $\epsilon > 0$; how will it play a spoilsport?

No matter what δ you choose, the definition of uniform continuity cannot be satisfied; that means for each choice of δ, there must be some point x that play spoilsport. And what is the meaning of play spoilsport? Well, x is close to y; but nevertheless F(x) and F(y) cannot be made closer than ϵ.

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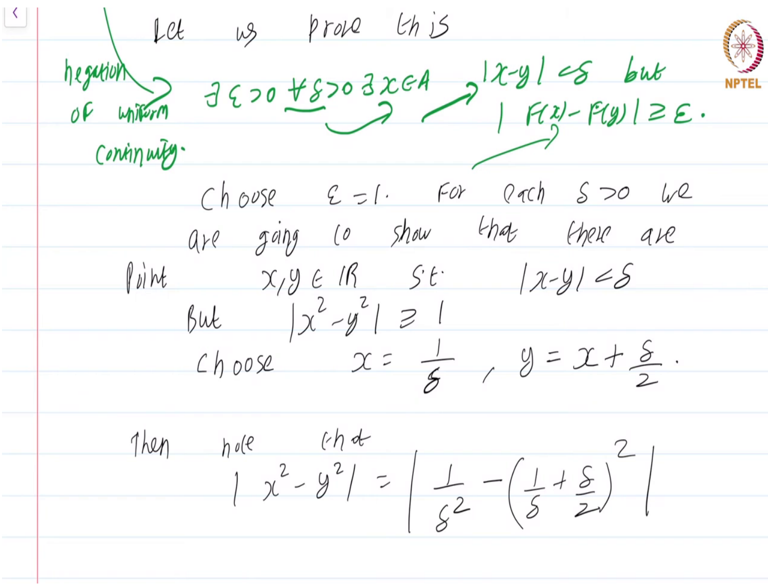

So, this is the negation, the negation of uniform continuity. How does this help us to show that x^2 is not uniformly continuous? Well, choose $\epsilon = 1$.

What we are going to do is, for each $\delta > 0$, we are going to show that that there are points $x, yin\mathbb{R}$, such that $|x - y| < \delta$, but $|x^2 - y^2| \geq 1$.

Now, what you do is, choose $x = \frac{1}{\delta}$ and $y = x + \frac{\delta}{2}$. Now, certainly $|x - y| < \delta$; in fact it is exactly equal to $\frac{\delta}{2}$. Then note that, $|x^2 - y^2|$ is in fact equal to just $|\frac{1}{\delta^2} - (\frac{1}{\delta} + \frac{\delta}{2})^2|$.

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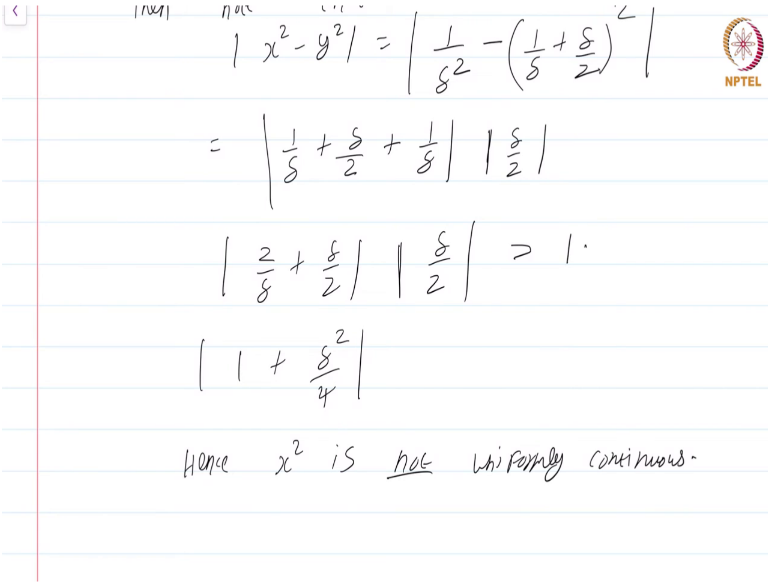

This is just $\frac{1}{\delta} + \frac{\delta}{2} + \frac{1}{\delta}$; I am just writing mod $|x + y||x - y|$ in a weird way ok, into $|\frac{\delta}{2}|$, right. So, you will get $|\frac{2}{\delta} + \frac{\delta}{2}||\frac{\delta}{2}|$, right.

So, when you multiply this out; it is clear that you get some quantity which is strictly greater than 1, right. In fact, what you get is $|1 + \frac{\delta^2}{4}|$, which is strictly greater than 1. So, we have contradicted the definition of uniform continuity. Hence, x^2 is not uniformly continuous.

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Proposition Let $F: A \to \mathbb{R}$ be a function. Then F is not uniformly continuous iff we can find sequence $x_n, y_n \in A$ with $x_n - y_n \to 0$ such that for some $\epsilon > 0$ $|F(x_n) - F(y_n)| \geq \epsilon$.

Theorem: Any continuous function $F: K \to \mathbb{R}$ where K is compact is uniformly continuous.

Proof:

This example should make this proposition obvious and I am going to leave it to you to prove it.

Let $F : A \longrightarrow \mathbb{R}$ be a function, then F is not uniformly continuous if and only if we can find sequences $x_n, y_n \in A$ with $x_n - y_n$ converging to 0, such that for some $\epsilon > 0$ $|F(x_n) - F(y_n)| \geq \epsilon$.

You can find two sequences that get arbitrarily close to each other, which is captured by saying $x_n - y_n$ converges to 0; but nevertheless $|F(x_n) - F(y_n)|$ is always somewhat far away, at least ϵ distance away.

So, we have seen example of a non uniformly continuous function. This example sort of says that; if you want to find a candidate point that allows you to defeat the $\epsilon - \delta$ definition for uniform continuity, you may have to go very far away.

We have fixed $\epsilon = 1$; if delta is very very small, then 1 by delta is very very large. So, to defeat the definition of uniform continuity, you have to show that no δ works for some choice of ϵ and you may have to go very far away.

This might suggest that if you have a function defined on $\mathbb{R}$, but you are restricted to some bounded piece; then the function is indeed going to be uniformly continuous, because you cannot arbitrarily choose points larger and larger away, if you are confined within a bounded area.

You should be thinking of the adjective compact and why it is called compact right now?

Theorem: Any continuous function $F: K \longrightarrow \mathbb{R}$, where K is compact, is uniformly continuous.

Why? Well, we have to prove it.

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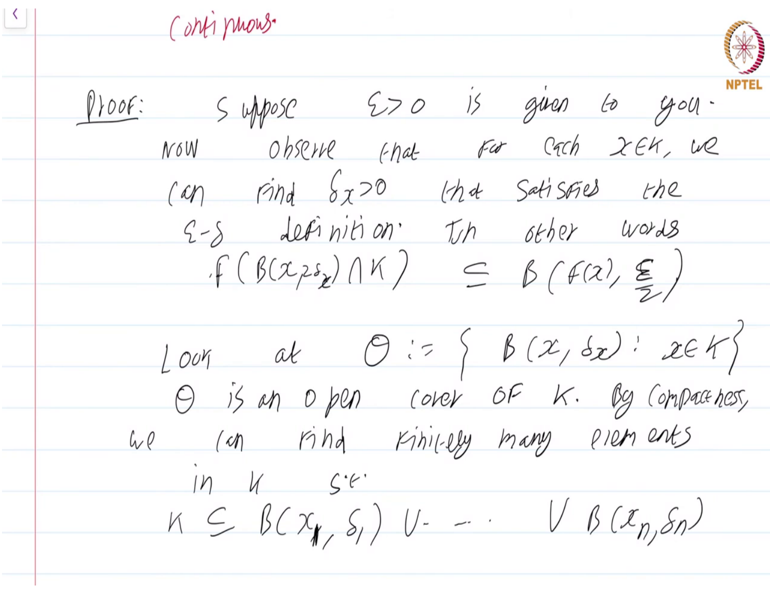

Proof: Now you can prove this in several ways; I am going to prove it using open covers and finite sub covers. Suppose $\epsilon > 0$ is given to you. I have to find a $\delta > 0$ that is independent of the point x in K; but depends only on ϵ, such that the epsilon delta definition is satisfied for every point in the set K.

Now, observe that for each x in K, we can find $\delta > 0$ that satisfies the $\epsilon - \delta$ definition right; that is because the function is given to be continuous. In other words, what I will do is; I will first change this notation slightly, I will make this δ_x.

In other words, $F(B(x, \delta_x)) \subset B(F(x), \epsilon)$; this is just the topological way of stating the $\epsilon - \delta$ definition, one of the equivalent ways of saying that a function F is continuous. Now, look at the collection

$\mathcal{O} := \{B(x, \delta_x) : x \in K\}$

$\mathcal{O}$ is obviously an open cover of K, right. So, one mistake I have made to be ultra precise; I cannot just take F of this, I have to do $B(x, \delta_x)$ intersect K right, otherwise it does not make sense. Let me make one more change with a slight tweak to the $\epsilon - \delta$ definition here, to make the rest of the proof flow more smoothly.

What I will do is, I will not just require $B(x, \delta_x)$ intersect K, F of that to be in $B(F(x), \epsilon)$; what I will do is? I will first put a by 2 here. So, that means, I will make it $\frac{\epsilon}{2}$ and here I will make it $2\delta_x$. So, all I have done is the $\epsilon - \delta$ definition is anyway satisfied; I have just adjusted my quantities, so that $B(x, 2\delta_x) \cap K \subset B(F(x), \frac{\epsilon}{2})$, this can always be done. Think about why if you are not sure.

Now, we know that this $B(x, \delta_x)$ is an open cover. By compactness, we can find finitely many elements in K, such that for ease of notation, I am going to simplify the subscripts, such that $B(x_1, \delta_1) \cup .. \cup B(x_n, \delta_n)$ this covers K. Technically I should be writing $\delta x_1, \delta x_2$ so on that is a bit cumbersome; I am just simplifying the notation, this is just by compactness.

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Choose

$\delta := \min\{\delta_1, \ldots, \delta_n\}.$

Claim: This δ works in the defn of uniform continuity.

Fix $x \in K$. Aim $F(B(x,\delta) \cap K) \subseteq B(F(x), \epsilon)$.

Since $x \in K$, we can find some x_i, say x_1, such that $x \in B(x_1, \delta_1)$.

Now, choose $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Claim is that this delta works in the definition of uniform continuity. Let us see why this is the case. So, fix x in K. Aim is to show

$F(B(x,\delta) \cap K \subset B(F(x),\epsilon)$, right. This is our aim; then we would be done. Well, since x is in K, we can find some x_i, say x_1 such that x is an element of $B(x_1,\delta_1)$.

So, these $B(x_i,\delta_i)$ is union of that is K. So, any element x will have to be in one of them; I am just taking for concreteness that particular element to be x_1, δ_1.

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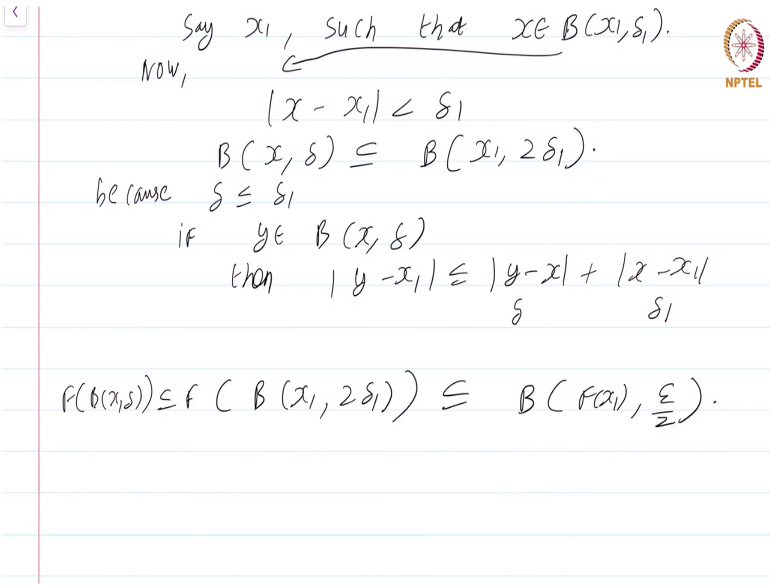

Now, this just means , $|x - x_1| < \delta_1$; but $B(x,\delta)$ will therefore be contained in $B(x_1,2\delta_1)$. This is because $\delta \leq \delta_1$. If you want to see a proof of this; let me just do it once, because I am going to pull a similar trick soon.

So, if $y \in B(x,\delta)$; then $|y - x_1| \leq |y - x| + |x - x_1| < \delta + \delta_1$ and we get this, this is just a simple application of the triangle inequality.

What does this give us, that gives us that $F(B(x_1,2\delta_1)) \subset B(f(x_1),\frac{\epsilon}{2})$. Why is that? Well, you can see that here; we had chosen with these additional constants that I added on later a factor of 2 on the left and a divided by 2 on the right precisely for this reason. You will get $F(B(x_1,2\delta_1)) \subset B(f(x_1),\frac{\epsilon}{2})$. In fact, you will get $F(B(x,\delta))$ is contained in this.

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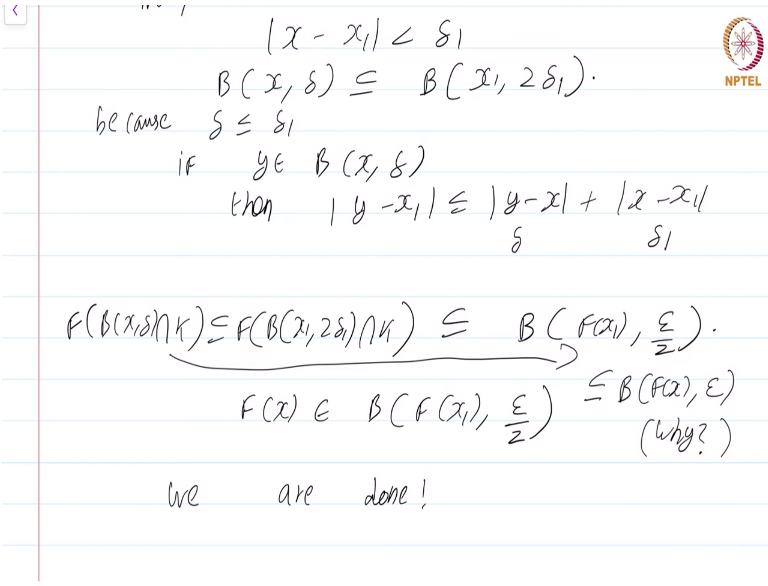

To be ultra-accurate, I must again make this minor change, $F(B(x_1, \delta) \cap K) \subset F(B(x_1, 2\delta_1) \cap K) \subset B(F(x_1), \frac{\epsilon}{2})$. How does this solve the problem for us? Well, F(x) is obviously there as an element of $B(F(x_1), \frac{\epsilon}{2})$ right; F(x) is certainly going to be an element of this. Well by simple logic, we see now that this thing is going to be contained in $B(F(x), \epsilon)$. Why? Well, it will be easy; we have already seen a similar argument.

So, what have we got? We have got $F(B(x, \delta) \cap K \subset B(F(x), \epsilon)$ we are done. So, this was a somewhat technical proof fiddling around with the quantities to make everything work; but it is just a straightforward application of the open cover form of compactness.

So, we have now shown that a continuous function on a compact set is uniformly continuous.

The question now arises, though this concept of uniform continuity, seems natural in the light that we are interested in when the function when the δ is a function of ϵ alone and does not depend on the choice of point, that is an interesting theoretical question.

But is it of any value? Well, look through the exercises or stay tuned for the chapter on integration, where we see a deeper application.

This is a course on real analysis and you have just watched the module on uniform continuity.