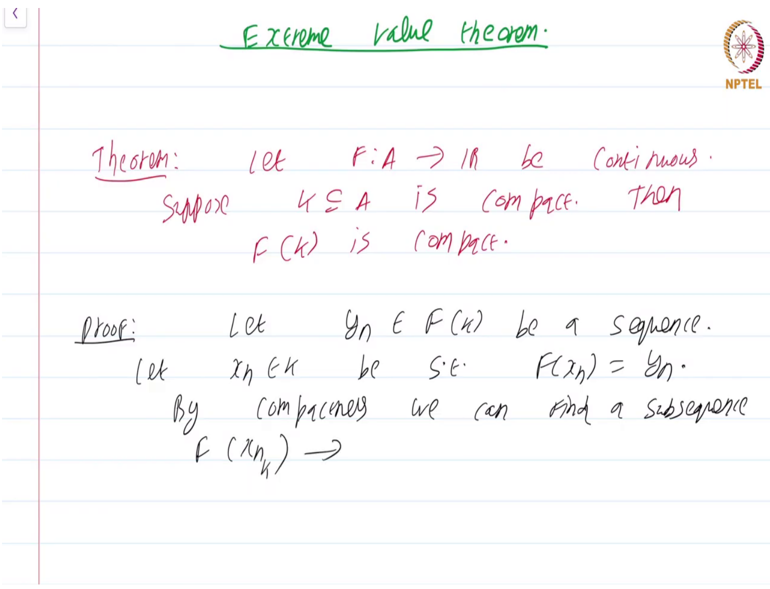
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 18.1
The Extreme Value Theorem

(Refer Slide Time: 00:14)

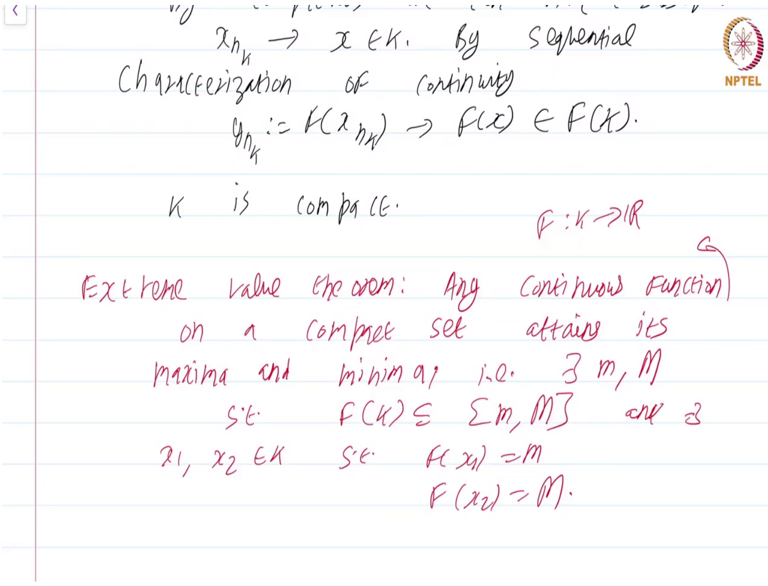


In this module, we are going to tie up the relationship between compactness and continuity. We are going to essentially study the interaction between these two concepts and derive several classical theorems that you might have learnt in calculus. Let me just phase state the first theorem which is the most important of the lot.

Theorem: Let $F : A \longrightarrow \mathbb{R}$ be continuous. Suppose $K \subset A$ is compact. Then F(K) is compact.

The proof is rather easy: let $y_n \in F(K)$ be a sequence. Let $x_n \in K$ be such that $F(x_n) = y_n$. For each y_n there will be at least one such x_n simply because y_n is there in the image of K under F. By compactness, we can find a subsequence x_{n_k} that converges to $x \in K$.

(Refer Slide Time: 02:05)



By sequential characterization of continuity, $F(x_{n_k})$ must converge to F(x), which is there in F(K).

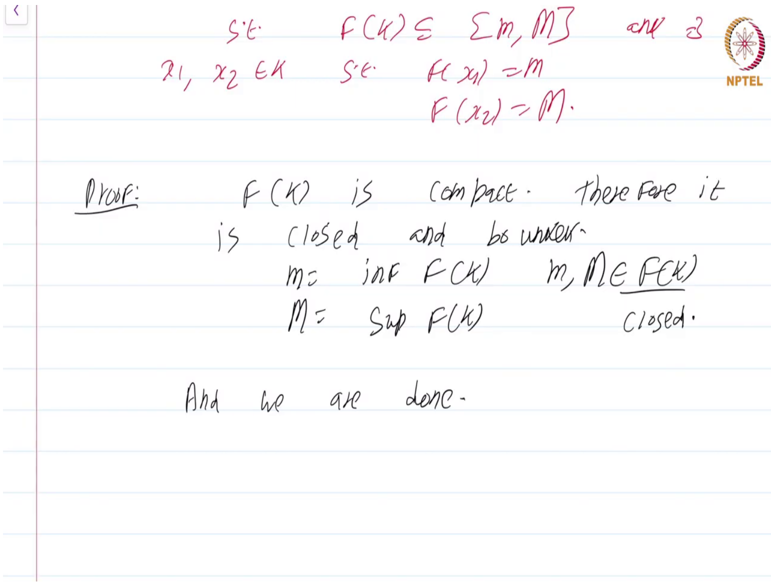
So, we started off with an arbitrary sequence $y_n \in F(K)$ and we have found a subsequence of that which we can just call y_{n_k} which converges to a point in F(K) which means K is compact. The proof was fairly easy.

Now, we get an immediate consequence of this which is the extreme value theorem.

Extreme value theorem: Any continuous function on a compact set attains its maxima and minima. That is, $\exists m, M$ such that $F(K) \subset [m, M]$ and $\exists x_1, x_2 \in K$ such that $F(x_1) = m,\ F(x_2) = M$.

This is an elaborate version of saying that a function on a compact set attains its maxima and minima.

(Refer Slide Time: 04:47)



Why is this an immediate consequence of the previous theorem? Well, think about it for a moment. F(K) is compact; therefore, it is closed and bounded. And now we can just see that $m = \inf F(K)$; and $M = \sup F(K)$. And $m, M \in F(K)$, because this is closed, And we are done.

We have shown that the infimum and supremum belong to F(K); therefore, there must be elements x_1 and x_2 in the set K such that $F(x_1) = m$, and $F(x_2) = M$. So, the extreme value theorem can be given a direct proof without resorting to any of this notion of compactness etc... But notice that we have got a much more general statement. This statement applies not only to closed intervals. It also applies to any compact set.

This is a course on Real Analysis. And you have just watched the module on the Extreme Value Theorem.