Real Analysis - I
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Lecture – 17.4
Equivalent Notions of Compactness

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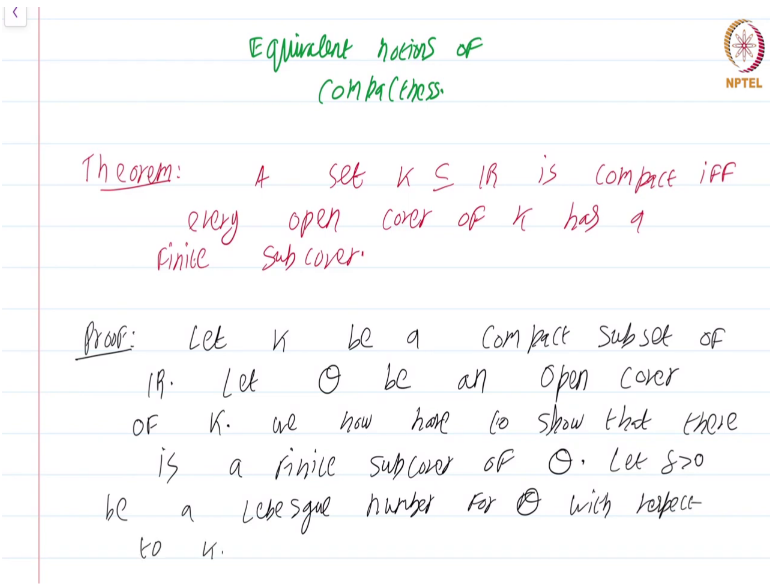

In this module, I shall characterize compactness in terms of open covers. So, without further ado, let me state the main theorem that we are interested in.

Theorem: A set $K \subset \mathbb{R}$ is compact if and only if every open cover of K has a finite subcover.

Let us prove this.

Proof: let K be a compact subset of $\mathbb{R}$. Let $\mathcal{O}$ be an open cover of K. We now have to show that there is a finite subcover of $\mathcal{O}$. Now, how do I show that there is a finite subcover of $\mathcal{O}$, I have to use this Lebesgue number property. We know that with respect to K, $\mathcal{O}$ has a Lebesgue number. So, let $\delta > 0$ be a Lebesgue number, Lebesgue number for $\mathcal{O}$ with respect to K.

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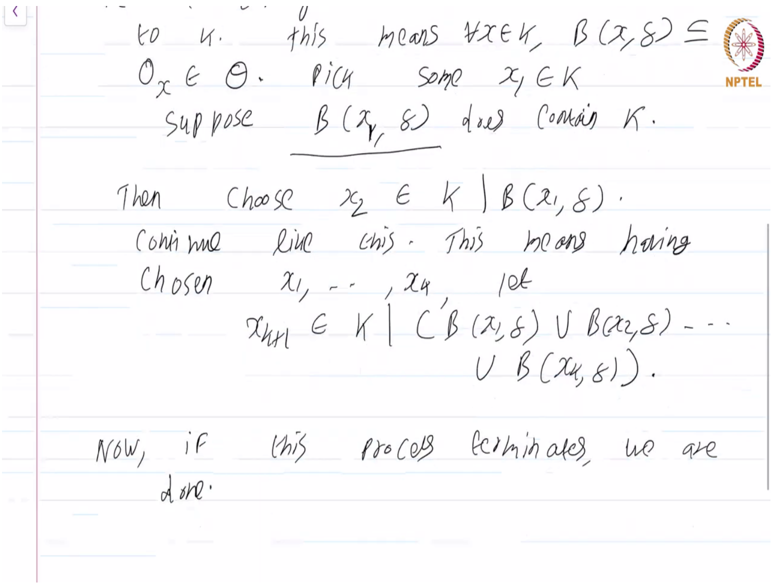

This means $\forall x \in K,\ B(x,\delta) \subset O_x \in \mathcal{O}$. This is the definition of a Lebesgue number. How does this help us? Well, pick some $x_1 \in K$. Suppose, $B(x_1,\delta)$ does not contain K. There are two possibilities: either this single open set $B(x_1,\delta)$, the single ball itself contains K or it does not.

We are assuming that this single ball does not contain K. Then choose $K \setminus B(x_1,\delta)$. Choose some point with K which is not there in $B(x_1,\delta)$. Continue like this. This means having chosen $x_1, \ldots, x_k$, let $x_{k+1} \in K \setminus B(x_1,\delta) \cup B(x_2,\delta) \cup \cup B(x_k,\delta)$..

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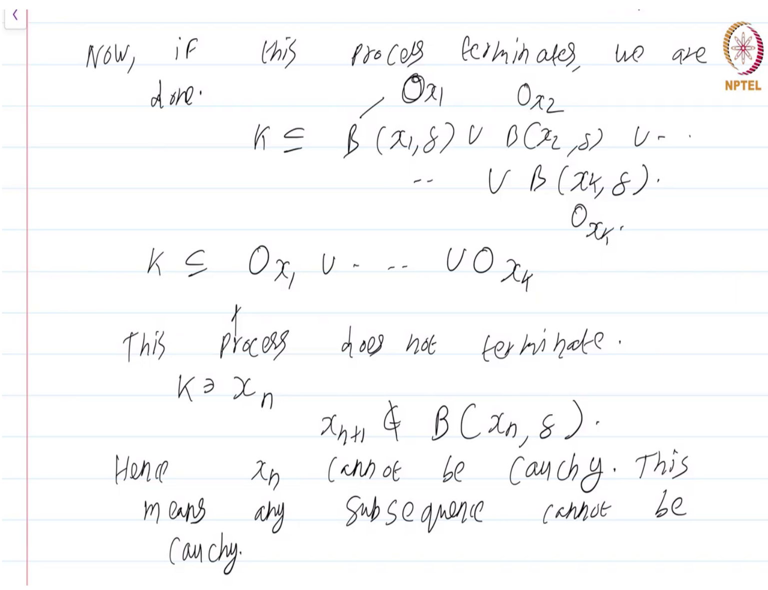

Now, if this process terminates, we are done. Why are we done if this process terminates? Well, the only way this process can terminate is if $K \subset B(x_1, \delta) \cup B(x_2, \delta) \cup . \cup B(x_k, \delta)$, that is some finite collection of open balls covering the whole of K. But each one of these open balls is contained in some element $O_{x_1}, O_{x_2}, \ldots, O_{x_k}$.

Remember δ is not an Lebesgue number, so that means, K will be a subset of $O_{x_1} \cup \cup O_{x_k}$, and we have found our required finite subcover. So, the only possibility is this process does not terminate, which just means we have found a sequence x_n, we have found a sequence $x_n \in K$.

Not only have we found an $x_n \in K$, this $x_{n+1} \notin B(x_n, \delta)$ right, that is exactly the way by which the next element in the sequence was chosen. Hence, x_n cannot be Cauchy right. For this choice of δ, x_{n+1} and x_n can never be δ close right by the very way by which we have chosen this x_n which means not only can x_n not be Cauchy this means any subsequence cannot be Cauchy as well.

Why is that the case? Well, it is because any given term of the sequence is at least δ distance away from all the prior terms. So, the same argument will tell you that this subsequence also cannot be Cauchy.

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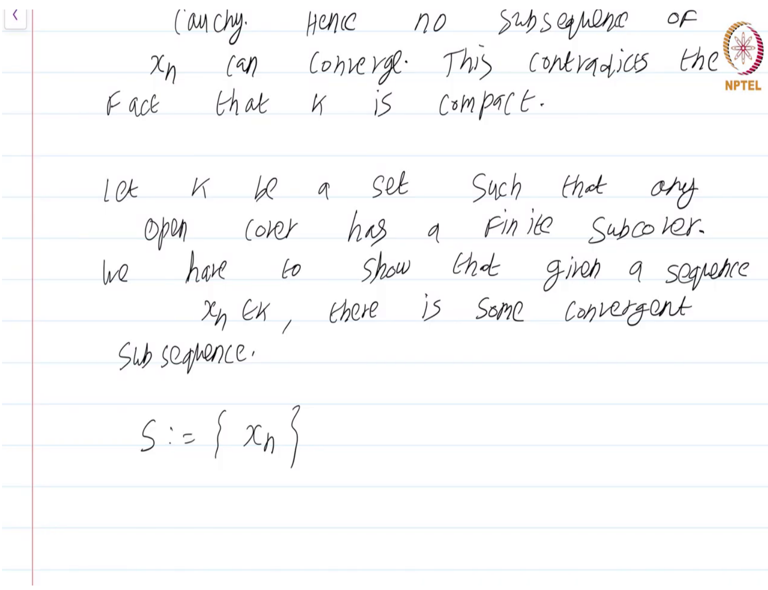

Hence, no subsequence of x_n can converge which contradicts the fact that K is compact right. Compact means any sequence has to have a convergent subsequence. We have just found a subsequence that does not converge. So, what we have shown is that if K is compact, any open cover will have a finite subcover.

Now, for the converse, let K be a set such that any open cover has a finite subcover . We have to show that given a sequence $x_n \in K$, there is some convergent subsequence . How does one do this? Well, this proof is a bit tricky. So, what I do is I collect together all the points x_n and put it as a set. Let me call this set S. The terms of the sequence, let me call it S.

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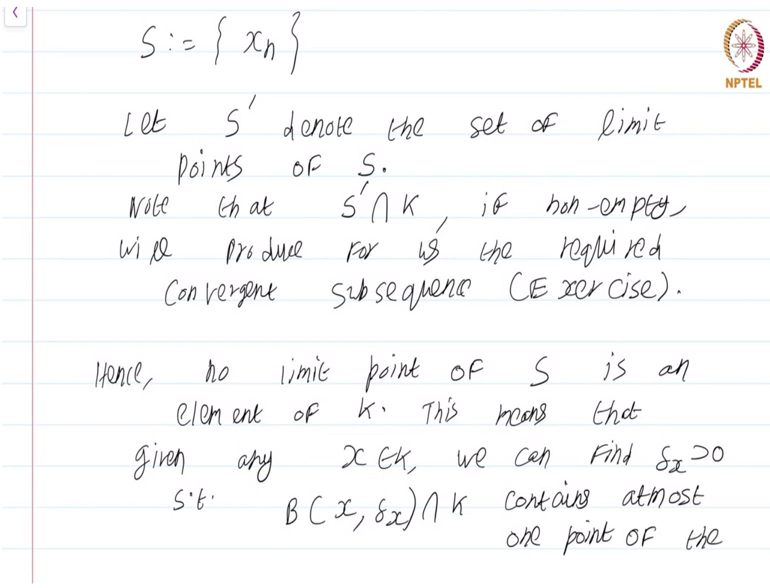

Now, let S' denote the set of limit points of S. Note that S' intersect K if non-empty will produce for us the required subsequence or rather required convergent subsequence. Now, this is left as an exercise for you.

What we are doing is our aim is to produce a convergent subsequence of x_n . We are looking at all the limit points of S. The claim is if one limit point of this set S is also there in K, this will allow us to produce a subsequence that is actually going to converge to a point of K which is what we want to show. Hence, no limit point of S is an element of K.

Now, what does this mean? This means that given any $x \in K$, we can find $\delta_x > 0$ such that $B(x, \delta_x) \cap K$ contains at most one point of the set S. Again exercise. So, this proof I am leaving two steps to you in the hope that you will get practice in thinking about the various concepts and the relationship between the various concepts.

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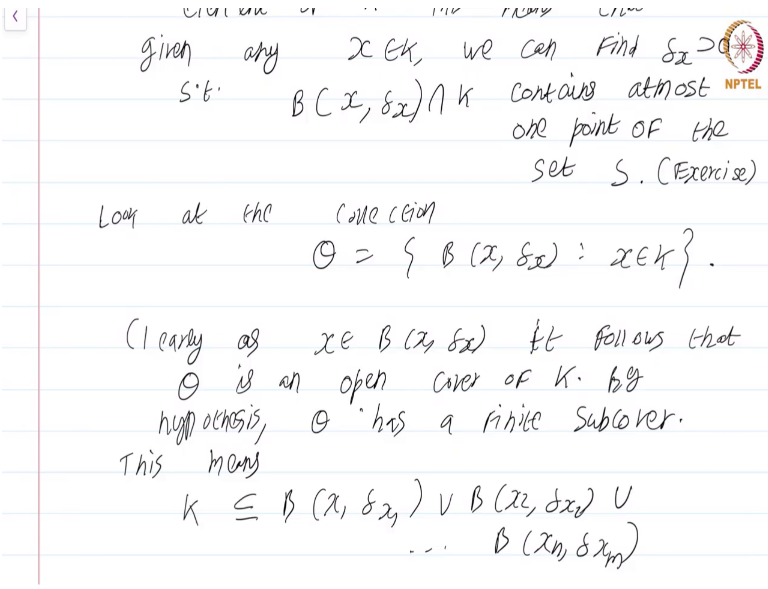

Now, let me give you a hint for this part because we are assuming that no limit point of S is an element of K that is the key .

Look at the collection $\mathcal{O} = \{B(x, \delta_x) : x \in K\}$, this is the collection of all such open balls.

Clearly, as $x \in B(x, \delta_x)$ which is no shock that it follows that $\mathcal{O}$ is an open cover of K. By hypothesis, $\mathcal{O}$ has a finite subcover. This means some $B(x_1, \delta) \cup B(x_2, \delta) \cup \cup B(x_m, \delta)$, this collection contains K.

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Clearly as $x \in B(x, \delta_x)$ it follows that $\mathcal{O}$ is an open cover of K. By hypothesis, $\mathcal{O}$ has a finite subcover. This means

$$K \subseteq B(x_1, \delta_{x_1}) \cup B(x_2, \delta_{x_2}) \cup \dots B(x_m, \delta_{x_m})$$

This forces the set S to be finite. Then it is easy to see that x_n has a constant subsequence. We are done.

This forces the set S to be finite , because each one of these balls $B(x_1, \delta), B(x_2, \delta), ., B(x_m, \delta)$ each one of them contains at most one point of S. So, this force of the set has to be finite, then it is easy to see that x_n has a constant subsequence. And we are done.

So, we have now characterized compactness in terms of open covers. Let me make the general remark that historically the proof that any closed and bounded subset of $\mathbb{R}$ for such a set, any open cover has a finite subcover is what was classically known as the Heine-Borel theorem. I have stated the what I call the Heine-Borel theorem slightly differently I have characterized compactness in terms of sequences which I believe is much better approach at such an elementary course.

This is a course on Real Analysis, and you have just watched the module on Equivalent Notions of Compactness.