Real Analysis - I

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Lecture – 17.2
The Heine-Borel Theorem

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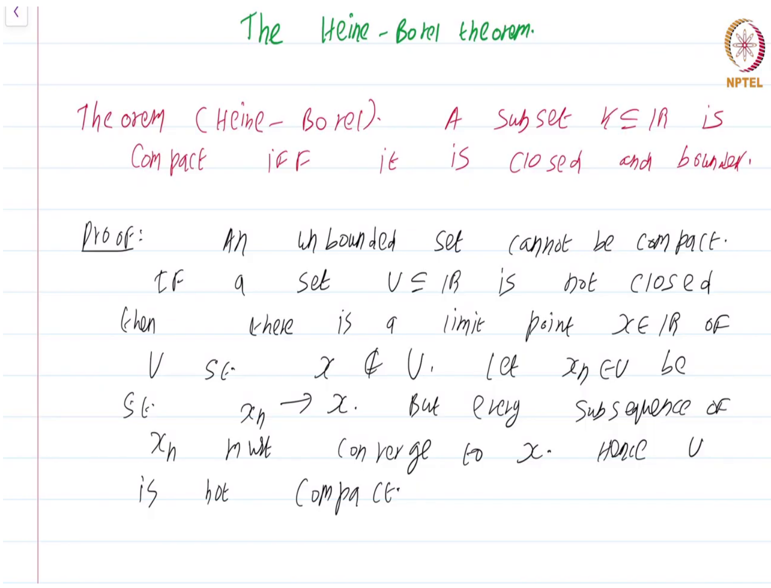

In this module, we shall give a complete description of every single compact subset of $\mathbb{R}$. Without further ado, let me state the theorem

Theorem (Heine-Borel) A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded.

And the proof is rather easy. An unbounded set cannot be compact, this we saw in the previous module as an example of non compact set. If a set $U \subset \mathbb{R}$ is not closed. In the last module one of the examples was that saying that any open set U would be non-compact – an open set cannot be non-compact.

Now, I am starting with if a set $U \subset \mathbb{R}$ is not closed, then it must immediately be not-compact right because a set that is not closed must be open. No, this is not true.

One of the common mistakes beginners make is to have this false dichotomy that set that is not open must be closed, and a set that is not closed must be open. The complement of a closed set is open, and the complement of an open set is closed does not mean that any set that is not open is closed and vice versa.

A subset of $\mathbb{R}$ just like a door can be open, closed or anywhere in between. So, please remember that. So, I cannot just say since U is not closed it must be open we are done. No, if a set U is not closed, then there is a limit point $x \in \mathbb{R}$ of U such that x is not in U that is the only way by which the set U can fail to be closed. It has some limit point which is not there in the set.

Let $x_n \in U$ be such that x_n converges to x. Such a sequence must exists simply because x is a limit point of U. But no subsequence or rather I will not want to have many negatives, but every subsequence of x_n must converge to x, hence U is not compact. The definition of compactness is that every sequence in that set has a subsequence that converges to a point in that set.

I have considered a point outside that set and found a sequence that converges to that point, no subsequence therefore can converge to a point inside that set. So, the definition of compactness is violated. So, we have proved one direction. We have shown that if a set is compact, then necessarily it has to be closed and bounded.

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is not compact.

let $K \subseteq \mathbb{R}$ be closed and bounded.
let $x_n \in K$ be any sequence. By Bolzano-Wierstrass, $x_{n_k} \to x \in \mathbb{R}$ for some subsequence. By closedness, $x \in K$ and we are done.

Cantor set.

Theorem (Nested intervals theorem for compact sets)
Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of nested intervals
$K_n \subseteq K_{n-1} \subseteq K_{n-2} \cdots \subseteq K_1$

Now, let $K \subset \mathbb{R}$ be closed and bounded. We have to show that it is compact. And the proof is exactly word for word what we did for a closed interval.

Let $x_n \in K$ be any sequence. By Bolzano- Weierstrass, $x_{n_k} \to x \in \mathbb{R}$ for some subsequence. By closedness x is in K and we are done. This is word for word the argument we gave for a closed interval. So, this short proof completely characterizes compact subsets of $\mathbb{R}$. They are precisely the closed and bounded sets of $\mathbb{R}$.

Now, there can be many exotic compact sets. We will explore one of them in a later module called the Cantor set. You must not be misled into believing that the only compact sets look somewhat similar to closed intervals or finitely many points, they can be very complicated. The cantor set is one exotic such object which we will discuss in some detail.

Now, the last part of the proof of the Heine-Borel theorem sort of suggests that even though compact sets could be very weird, they do behave very much like closed intervals in a lot of senses. This analogy that closed sets, closed and bounded sets are similar to closed intervals can be illustrated in another theorem

Theorem(Nested intervals theorem for compact sets.)

Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of nested intervals, this just means that $K_n \subset K_{n-1} \subset K_{n-2} \subset .. \subset K_1$.

The sets are all nested inside each other like a Russian doll.

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Theorem (Nested intervals theorem for compact sets)

Let $\{K_n : n \in \mathbb{N}\}$ be a sequence of nested compact non-empty sets.

$$K_n \subseteq K_{n-1} \subseteq K_{n-2} \cdots \subseteq K_1$$

Then $\bigcap K_n \neq \emptyset$.

<u>Proof:</u> Let $x_n \in K_n$. Note that

$$\{x_n : n \geq m\} \subseteq K_m.$$

$x_n \in K_1 \quad \forall n.$

By compactness, some subsequence of

Suppose, no other assumption is needed, be a sequence of nested compact non-empty sets, I do not want any of these sets to be empty. Then, $\cap K_n \neq \phi$.

In fact, $\cap K_n$ would be, I will talk about that later. Let me just finish off with this $\cap K_n$ is non-empty.

Proof: What I do is the following. Let $x_n \in K_n$. Since each one of these sets is non-empty by assumption by a hypothesis, I can find an element $x_n \in K_n$ right. Let $x_n \in K_n$. Note that $\{x_n : n \geq m\} \subset K_m$. Why is this ? Because these are nested intervals.

So, in particular, this entire $x_n \in K_1 \; \forall n$. So, all the terms of the sequence beyond the mth term will be a subset of K_m . Now, by compactness some subsequence of x_n converges to $x \in K_1$.

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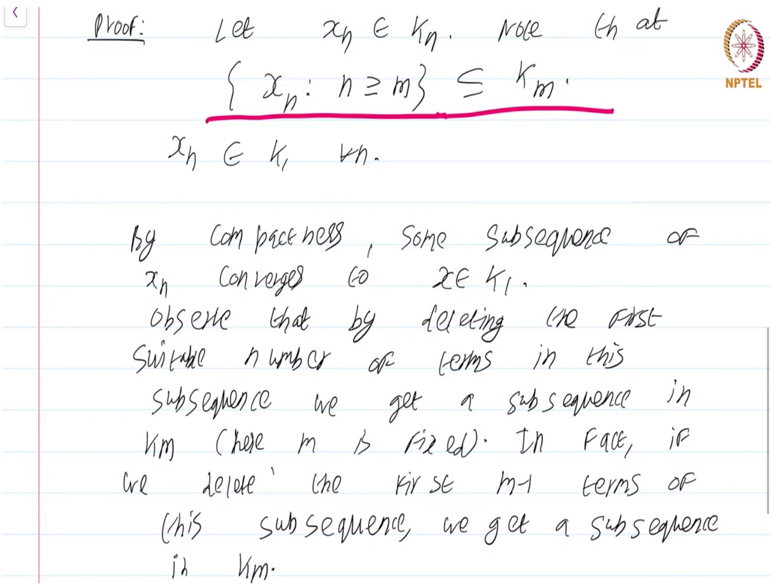

Now, observe that by deleting the first suitable number of terms in the subsequence, we get a subsequence in K_m , here m is fixed that subsequence will depend on m. In fact if we delete the first (m - 1) terms of the subsequence, we get a subsequence in K_m . So, at the most the first m terms can be (m - 1) terms can be outside of K_m .

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Observe that by deleting the first suitable number of terms in this subsequence we get a subsequence in K_m (here m is fixed). In fact, if we delete the first $m-1$ terms of this subsequence, we get a subsequence in K_m. This means this new subsequence also converges to x. Therefore, $x \in K_m$ (by compactness). This means

$$x \in \bigcap K_n \text{ and we are done.}$$

So, this means, this new subsequence also converges to x. I have just deleted the first few terms of a subsequence, the resultant subsequence should also converge to x, therefore, $x \in K_m$ again by compactness.

Within a compact set, if you have a sequence that converges that point to which it converges must be in K right, so that means, $x \in \cap K_n$ and we are done. So, this nested intervals property generalizes to a nested intersection theorem for compact sets.

This is a course on real analysis. And you have just watched the module on the Heine-Borel theorem.