Real Analysis - I
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Lecture - 2.2
Basic Logic

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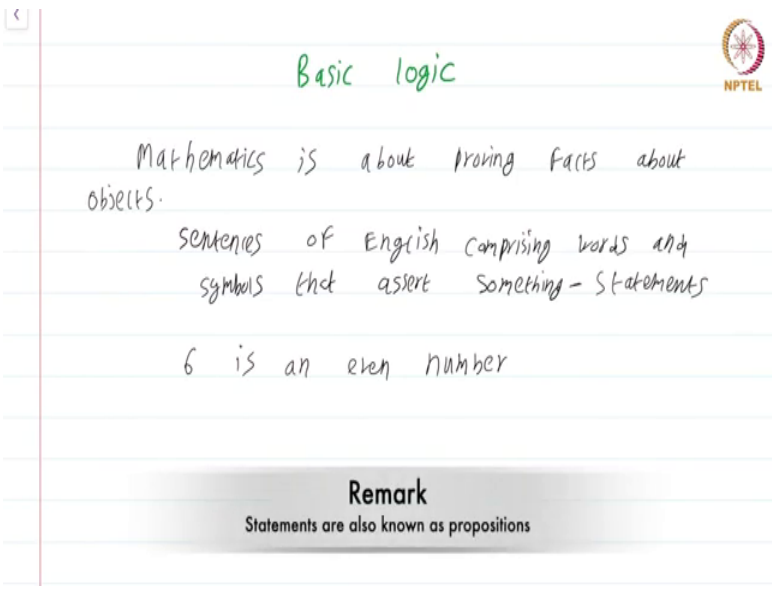

In this module, we will discuss elements of logic. Now as I have talked about before, Mathematics is about proving properties or facts about objects. So, mathematics deals with certain sentences of English and comprises words and symbols that assert something.

So, rather than this wordy definition, seeing a few examples, we clarify it most aptly.

6 is an even number. This is a sentence of English. It asserts something about an object the number 6 which is a mathematical object. It is asserting that 6 is an even number. So, this is a sentence that is asserting something such sentences are called statements. These are called statements.

Sentences that have a true-false assertion are called statements. So, mathematics is essentially about proving facts about certain objects; that means, mathematics is about determining the truth or falsity of statements that deal with mathematical objects. So, this happens to be a true statement as we all know.

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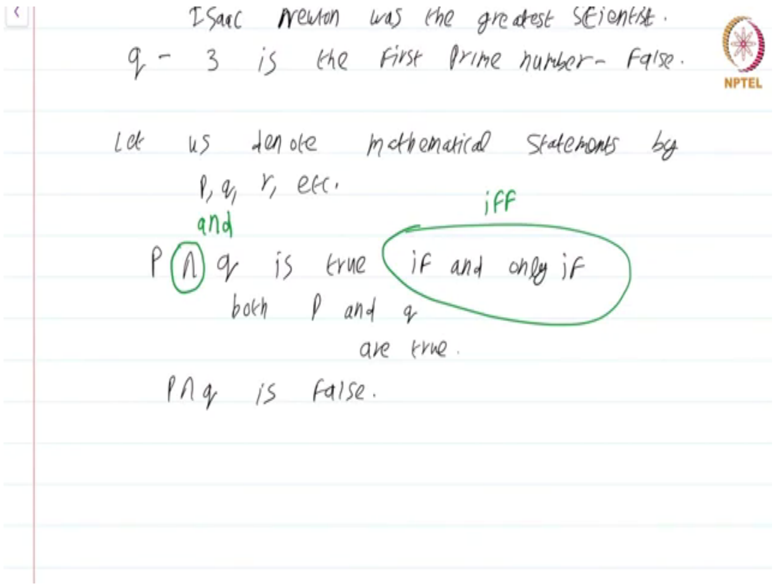

Now, consider another statement ‘Isaac Newton was the greatest scientist’. Now this is a sentence that certainly asserts something, but what it asserts is not really about a mathematical object unless somebody comes on and defines what Isaac Newton and greatest mean mathematically. It will not clear whether this is true or false. So, mathematics is not concerned with such statements. Mathematics is about mathematical objects.

Now, you might ask well why is 6 an even number? That is a fair question. Well the thing is mathematics essentially defines what 6 is, defines what even number is, then it is the goal of the mathematician to show that 6 is an even number. How one does this, how one defines objects and how one manipulates these definitions to prove interesting theorems is using logic. That is what this module is about.

So, this second statement even though it asserts something is not a mathematical statement. So, mathematics will not say true or false about this. But this is a false statement, 3 is the first prime number. This is a false statement.

So, to determine the truth or falsity of a statement, we need to first develop a language for defining the objects, then develop a language for manipulating these objects and proving stuff. So, let us denote mathematical statements by p, q, r etc. So, we will denote, we will take a more symbolic approach, p, q, r are supposed to denote mathematical statements. They could be 6 is an even number which is a true mathematical statement or 3 is the first prime number which is false.

First, we need a language for combining statements and that is provided by certain keywords. We have $p \wedge q$, this is one way of combining two mathematical statements. As you can guess, this compound statement $p \wedge q$ this is true if and only if. Now this combination of words if and only if arises so frequently in mathematics that the famous mathematician and textbook writer and expositor Paul Halmos developed a shortcut which we will borrow if and only if as iff .$p \wedge q$, this is one way of combining two mathematical statements. As you can guess, this compound statement $p \wedge q$ this is true if and only if. Now this combination of words if and only if arises so frequently in mathematics that the famous mathematician and textbook writer and expositor Paul Halmos developed a shortcut which we will borrow if and only if as iff .

So, statement $p \wedge q$ is true, if and only if both p and q are true. So, the statement, if you call a statement 6 is an even number; if you call this statement p and if you call the statement q, $p \wedge q$ is false because p is certainly an even number , p is true, but q which asserts that 3 is the first prime number is false .$p \wedge q$ is true, if and only if both p and q are true. So, the statement, if you call a statement 6 is an even number; if you call this statement p and if you call the statement q , $p \wedge q$ is false because p is certainly an even number , p is true, but q which asserts that 3 is the first prime number is false .

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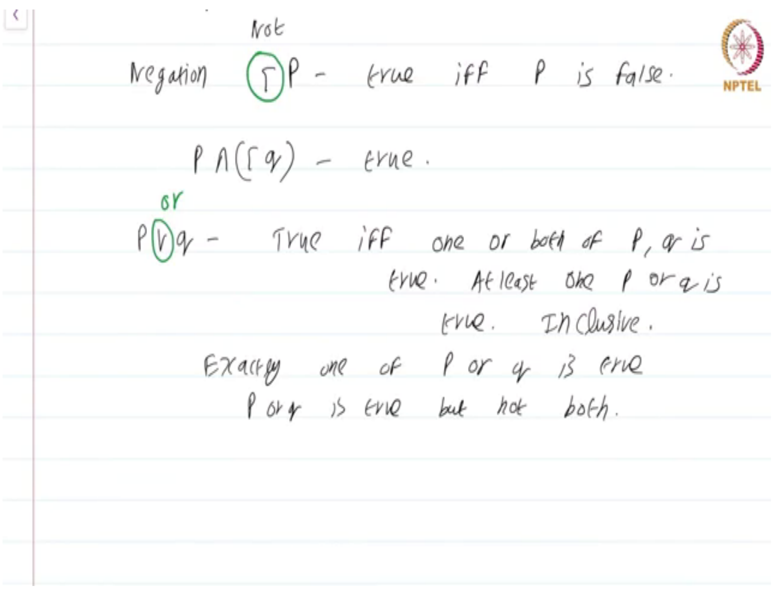

Now, we would like to assert that p is true, but q is not true. We want to combine these two in an easy way, for that we can use the negation also denoted $\neg p$ like this. This is true if and only if p is false . So, the statement $p \wedge \neg q$, this is a true statement because the statement p asserts that 6 is even, the statement q asserts that 3 is the first prime, 3 is not the first prime so negation q is true. So, $p \wedge \neg q$ is true . So, the negation which is also called $\neg p$, this is read not p .

Notice that we had used and in the definition of the intersection of two sets. So, there the 'and' is the mathematical $\wedge$ that I am defining here. Now the third operation that is needed is $p \vee q$. This is read 'or'. This is a statement that is true if and only if one or both of p , q is true; that means, at least one of p or q is true. Notice that the mathematical or is always inclusive. Even if both things are true, the statement p or q is still true.

In English, sometimes we use inclusive or, sometimes we use exclusive or. For instance, if I say I will go to the movies or I will go and watch a cricket match, it is not; it is not assumed that both can happen. In fact, both cannot happen. You cannot go to the movie and watch the cricket match simultaneously, unless the theatre happens to be telecasting the cricket match as a movie.

So, $\vee$ is always inclusive in mathematics. If we want to use exclusive 'or' we will say exactly one of p or q is true, we will or we will say p or q is true, but not both. So, we will be explicit when we are implying, when we want to suggest that we are using the exclusive 'or' .

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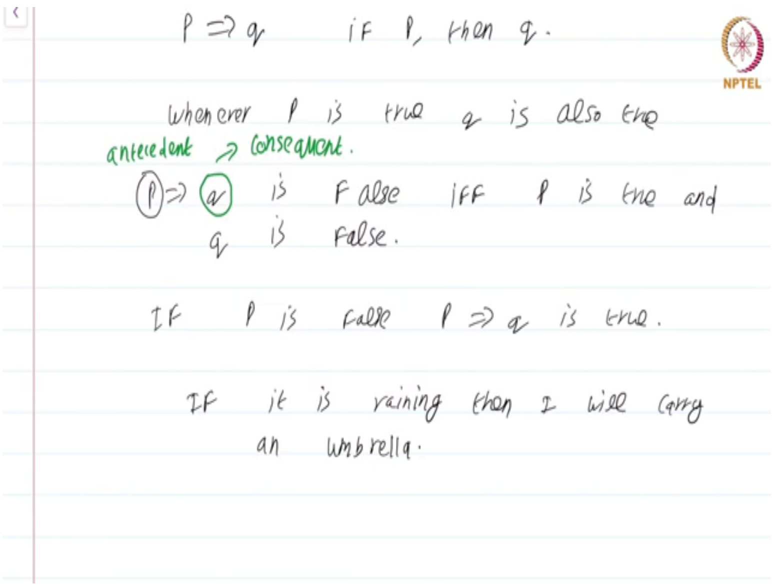

Now, comes the complicated and the most important thing implications $p \Rightarrow q$, if p , then q . Recall that we had also used if p , then q somewhere in the previous module on set theory, please try to figure out where it is that we used . What does this mean? This says that whenever p is true, q is also true.

Now, the best way to figure out what this implication means is to tell you when it is false. So, $p \Rightarrow q$ is false all the others I told you when it is true, now I am going to tell you when it is false. $p \Rightarrow q$ is false if and only if p is true and q is false. This is the only scenario in which $p \Rightarrow q$ is actually false.

The statement p must be true, but the statement q must be false. So, the statement p is also called the antecedent, whereas the statement q is called the consequent. So, there are several ways to read this q follows from p , if p then q , p implies q .

Now, you might ask what will happen if p is false. If p is false , then $p \Rightarrow q$ is true. Why is this? Well of course, it follows from the very definition. If the statement is false, $p \Rightarrow q$ is false only under the scenario where p is true and q is false, but why would you define it this way?

We have full flexibility of defining implication in whatever way we want. Why did we define it this way? Well there are several mathematical reasons for this which you will realize in the course of your studies in mathematics, but let us take a real world example to understand what this means.

Suppose I say something like if it is raining, then I will carry an umbrella. What is this statement really saying? The statement is really saying that whenever there is rain, I am going to be carrying an umbrella. Suppose it is not raining, then this statement gives you no data about my behaviour.

I might still be carrying an umbrella, I might not be carrying an umbrella irrespective of which situation it is, whenever it is not raining. This statement is not really giving you any data about my behaviour. Therefore, this statement will be true whenever it is not raining irrespective of whether I am carrying an umbrella, or I am not carrying an umbrella. This statement is not really making any assertion about that situation and such things arise frequently in mathematics.

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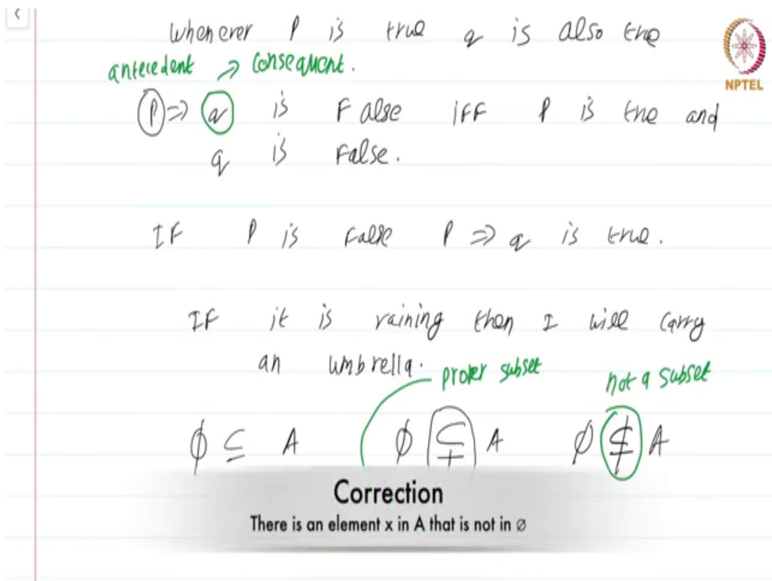

Well let us take the empty set. I claim that the empty set is a subset of any set A whatsoever. This is now an appropriate time to introduce some notation about subsets. We have already seen the meaning of subsets. Now, I want to introduce first the notion of a proper subset which is written with a subset symbol and a horizontal line cut vertically just the horizontal line ($\subsetneq$) and ϕ not a subset of A , which is written with a cross fully across the subset symbol ($\phi \nsubseteq A$).

The first proper subset means that the set is $\{\phi\}$ is contained in the set A whereas, the second one means that it is not a subset at all; that means, there is some element $x \in \phi$ which is not an element of A. Now, let me clarify further what the proper subset means not only is $\phi \subseteq A$, but there is some element $x \in \phi$ which is not there in A. That means, ϕ is a subset of A, but ϕ is not equal to A. That is what proper subset means.

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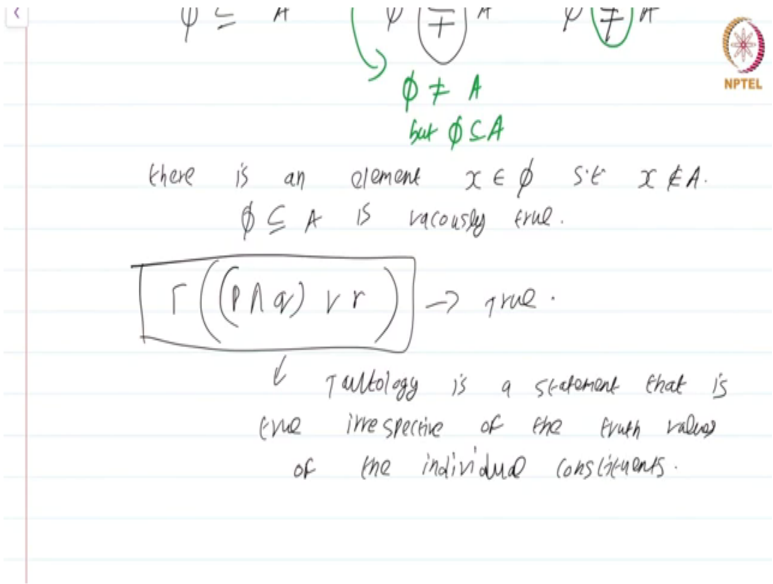

Now, we can use the meaning of if-then, try and prove when ϕ will be a subset of every set A. If you look at the definition of if-then, then the only possibility is there is an element $x \in \phi$ such that $x \notin A$.

Only under this scenario will phi fail to be a subset of A. But there is no element x in the empty set. The empty set is by definition a set that has no elements. So, the statement $\phi \subseteq A$ is vacuously true, that means, it is an implication where the antecedent is false. So,$\phi \subseteq A$.

Now, we will repeatedly use vacuous arguments. If this is a bit puzzling, trust me that you will by practice it will be second nature to you. So, we have defined the basic operations of logic.

Now, how do you prove something about statements well, if you have a very complicated statement let us say $\neg((p \wedge q) \vee r)$, how do you prove something like this? How do you check whether such a statement is true? Well, what does it mean for such a statement to be true?

What it means is that irrespective of whether p is true or false, q is true or false or r is true or false this statement should be true. Then, we can say that this is a true statement irrespective of what p , q and r what truth values they take. Such a statement is called a tautology.

Tautology is a statement that is true irrespective of the truth values of the individual constituents. So, how do you show that something is a tautology? Well, you are all familiar with truth tables. I will not really bother explaining in great detail, but let me just illustrate with a simple example.

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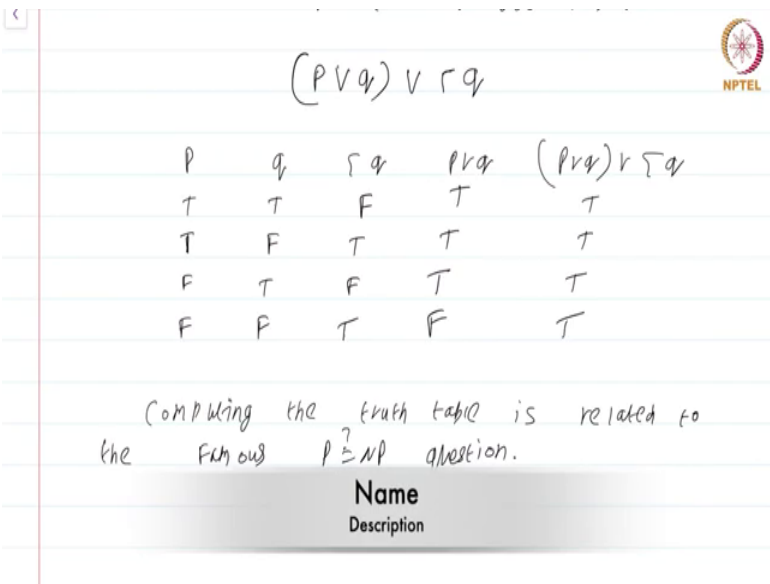

Let us look at the statement $(p \vee q) \vee \neg q$. Look at this statement $(p \vee q) \vee \neg q$ (p or q or negation q). I claim that this statement is true always irrespective of the individual assignment of truth values for p and q .

Well how do you check this? You draw a table $p, q, \neg q, p \vee q, (p \vee q) \vee \neg q$. Well, true true, true, false, false ok, true, true, true, false, false, true, false, false. So, in the first two rows, I have enumerated all the possible combinations of truth values that are possible for p and q .

Now I can compute what $\neg q$ is. If q is true, $\neg q$ is false, true, false, true, $p \vee q$ is true here, true here, true here, false here . Now, $(p \vee q) \vee \neg q$ is true, true, true, true. So, irrespective of how you assign the truth values to p and q , you will always get the final for $(p \vee q) \vee \neg q$, it is always true, such a thing is called a tautology.

A tautology is therefore, a statement that is true irrespective of the truth values of the constituent components like p , q and r. Rather, a tautology is true because of the meanings of the logical operators that occur in the statement like or, and, negation and implies.

You will notice that if a statement has just two letters p and q , then the truth table will have 4 lines that is 2^2 entries. If there are n letters p , q , r, s and so on, then there will be 2^n entries in the truth table for such a statement.

Therefore, determining whether a statement is a tautology or not, by using a truth table is highly inefficient. Nevertheless, it is an interesting programming exercise if you know programming to implement a program that takes as input a proposition from the user and computes and displays the truth table.

I will just mention that computing the truth table is related to the famous $P = NP$ question. So, if you are not familiar with this, this has nothing to do with the course. This is just a diversion. If you are familiar, please I urge you to explore further that being said mathematics usually does not deal with tautologies.

Let me illustrate by what I mean by this,

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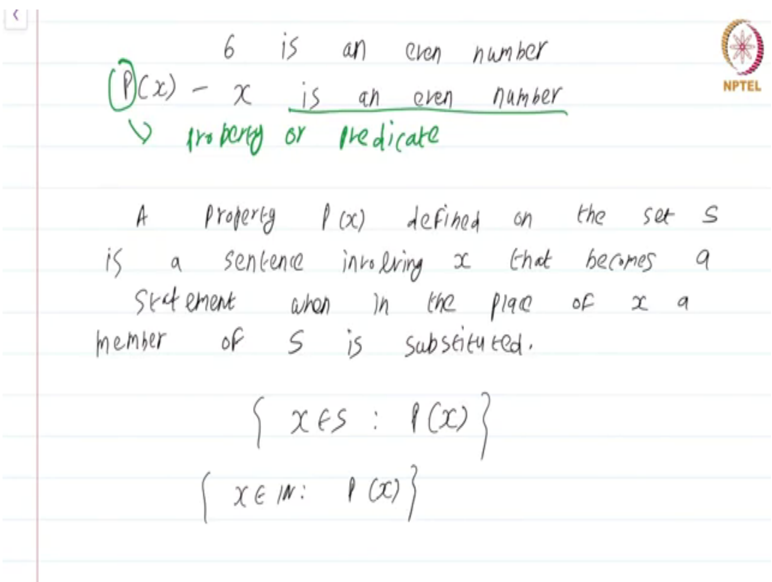

Look at the statement 6 is an even number. This is a statement that we had considered. Now instead if I write x is an even number, this is no longer a statement. This is not a statement because this is not a true or false question whether this statement is true or not depends on what x is, but such statements abound in mathematics.

So, we will have a notation for this, we will call it $P(x)$. Such a P is what is known as a property or predicate , I will stick to the term, property. It is more intuitive than the term predicate, but many logic textbooks use the term predicate.

A predicate or property $P(x)$ defined on the set S is a sentence involving x that becomes a statement when in the place of x a member x of S is substituted. So, a predicate is a sentence that by itself is not a statement there is a placeholder x.

In the place of x, you can substitute members of a set S once you do that, you get a statement. So, being even is not really a sentence that is a statement, it is a property, being an even number is a property. This property is enjoyed by some subset of natural numbers .

So, if you recall in a previous module, we had discussed defining sets using properties. Now, I think it will be clear what that definition means. The set $\{x \in S : p(x)\}$ is merely the subset of S for which the property when substituted with x which becomes a statement is true. Examples include the set of even numbers. We can call this property $p(x)$ like we have done and just say $x \in \mathbb{N}$ such that $p(x)$, this will give you all the even numbers .

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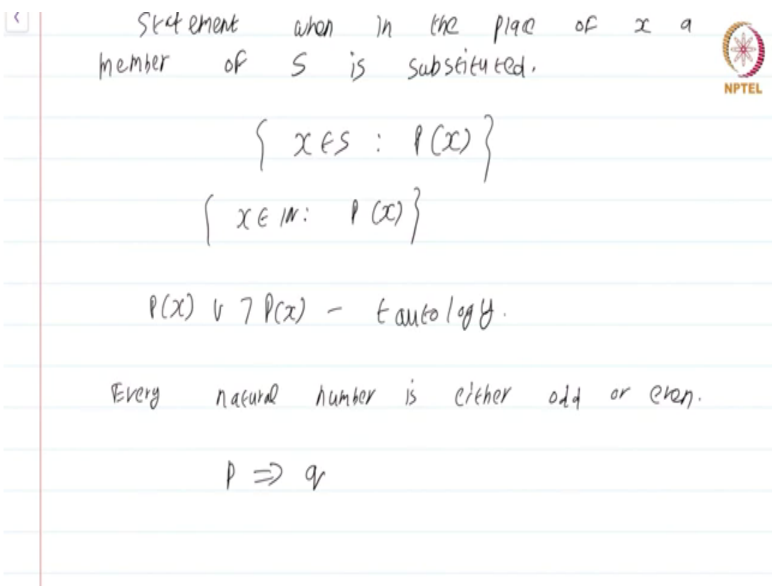

Now, I said that in mathematics, tautologies are usually rare. Let me illustrate now, having defined what a property is let p again be the property that it is an even number, x is $p(x)$ be the property that x is an even number. Observe that the statement $p(x) \vee \neg p(x)$ is a tautology.

This is a tautology irrespective of what you substitute for x. It is a tautology irrespective that this $p(x)$ could have been an entirely different property, it is still going to be a tautology because a tautology does not depend on the nature of the individual letters that constitutes the statement, but rather it depends only on the meaning of the logical operators either a statement is true or not. Therefore, $p(x) \vee \neg p(x)$ is a tautology.

So, we have proved the grand theorem that every natural number is either odd or even. An odd number is just the number that is not an even number. Under these definitions, we have shown that every natural number is either odd or even. Now you will agree with me that such a statement is completely trivial and there is no point in proving such statements.

In fact, if mathematics is merely about proving that statements then mathematics is a futile exercise. But statements in mathematics are rarely of this type. They are usually not tautologies, they are usually implications. They are statements of the type $p \Rightarrow q$. You have to establish $p \Rightarrow q$, rather you will have to provide a proof.

So, in the next module, I will talk more in detail about proofs.