Real Analysis - I

Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 16.3
One Sided Limits

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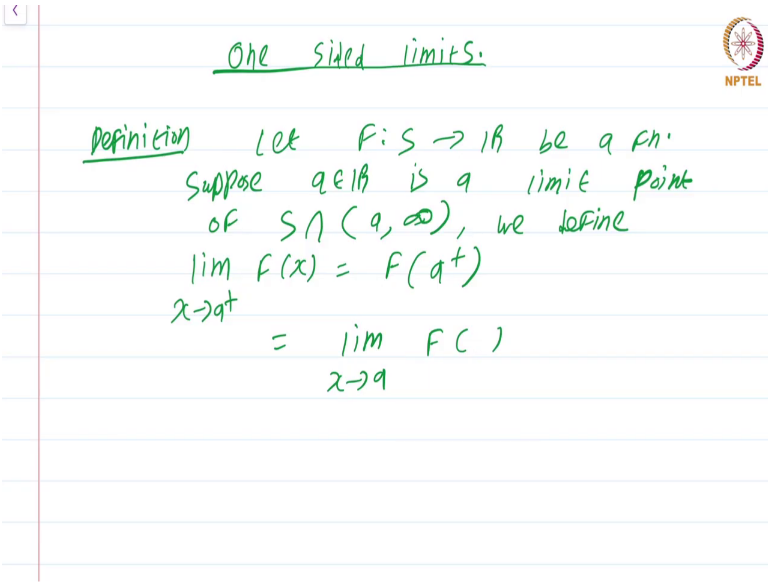

We now define One-Sided Limits which is a very useful concept especially for checking continuity at a point. Since our definition of limit was quite general, we can make quick work out of the definition of one-sided limit. So, without further ado definition.

Definition: Let $F : S \longrightarrow \mathbb{R}$ be a function. Suppose $a \in \mathbb{R}$ is a limit point of $S \cap (a, \infty)$. We define $\lim_{x \to a^+} F(x)$, this is also denoted $F(a^+)$ in many textbooks to be $\lim_{x \to a} F(x)$, but not of the function F.

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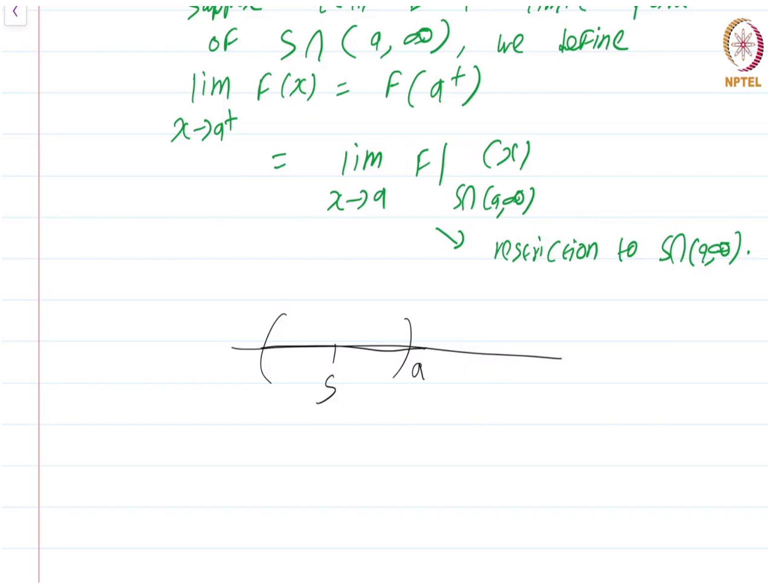

What we do is we restrict this function to $S \cap (a, \infty)$ and then take limits. Recall this stands for restriction to $S \cap (a, \infty)$. So, again a picture is worth a thousand proofs. Here it is fairly simple. You have some set S this is supposed to be the set S and you are focusing on a point a. Let us say a is here. You want to define so, if a is there will be a problem. So, what we will do is we will draw a better picture.

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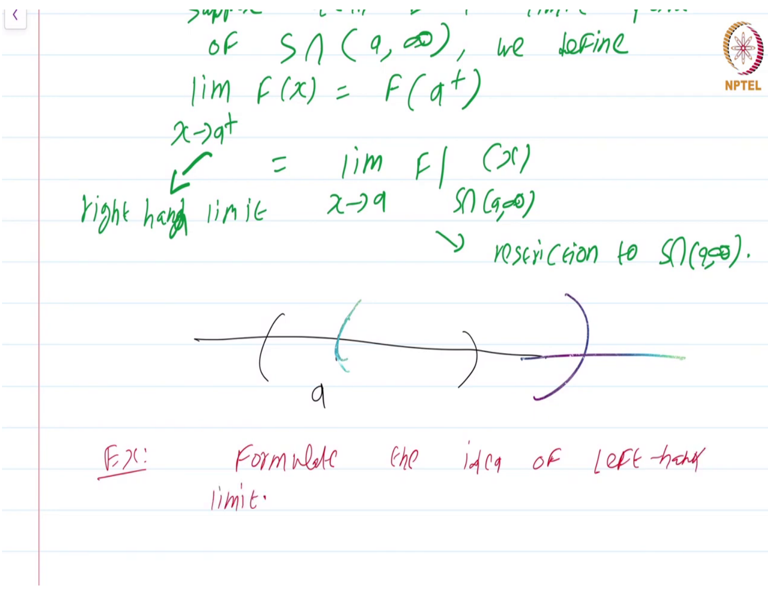

We have the set S, and we are taking the point a here. Then, to define the right-hand limit at this point a which is denoted by $\lim_{x \to a^+} F(x)$ so, let me just call this a right-hand limit. To define the right-hand limit at the point a, what you do is you focus on this portion; this portion of the set, restrict F to this portion, then you take the usual limit that you are familiar with.

So, because we required only the point a to be an adherent point of a given set to define limit at that point not at an adherent point, we required only the point to be a limit point of the particular set our earlier definition of limit can be borrowed and used in this case also. So, we get a quick definition of the right-hand limit.

Exercise formulate the idea of left-hand limit; of left-hand limit.

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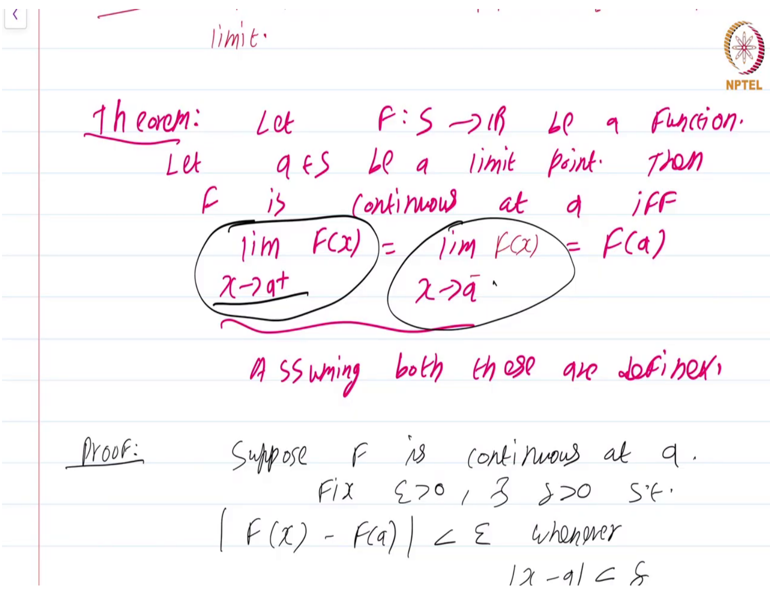

Now, let us prove one interesting theorem and this is a theorem that is sort of taken to be a definition in high school.

Theorem: Let $F : S \longrightarrow \mathbb{R}$ be a function. Let $a \in S$ be a limit point. Then, F is continuous at a if and only if $\lim_{x \to a^+} F(x) = \lim_{x \to a^-} F(x) = F(a)$. Assuming both these are defined.

Now, it can happen that a is not a limit point of $S \cap (a, \infty)$ or it is not a limit point of $S \cap (-\infty, a)$ that can happen. So, I am assuming both of these are defined at the point a, then the function is continuous if and only if the left-hand limit is equal to the right-hand limit is equal to the functional value.

Proof: Suppose F is continuous at a. Then, what this says is fix $\epsilon > 0$, then this just says $|F(x) - F(a)| < \epsilon$, there exist $\delta > 0$ such that whenever $|x - a| < \delta$ and $x \in S$.

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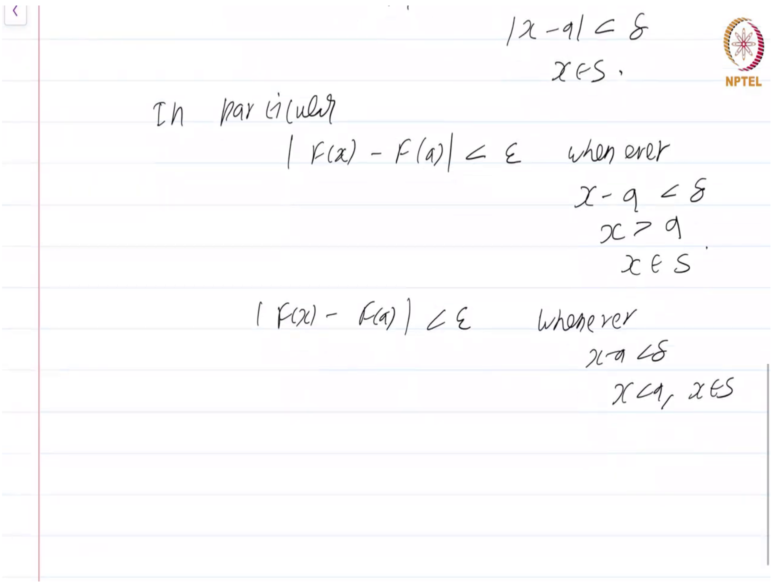

In particular $|F(x) - F(a)| < \epsilon$ whenever $x - a < \delta$ and x > a and $x \in S$. I am just taking a special case; I am just looking at those points x which lie to the right of 'a' by declaring that x > a .

In particular, this is true and so is the fact that $|F(x) - F(a)| < \epsilon$ whenever $x - a < \delta$ and x < a, $x \in S$, this is also true. Both will have to be true if you want the previous statement $|F(x) - F(a)| < \epsilon$ whenever $|x - a|, \delta$ and $x \in S$ right.

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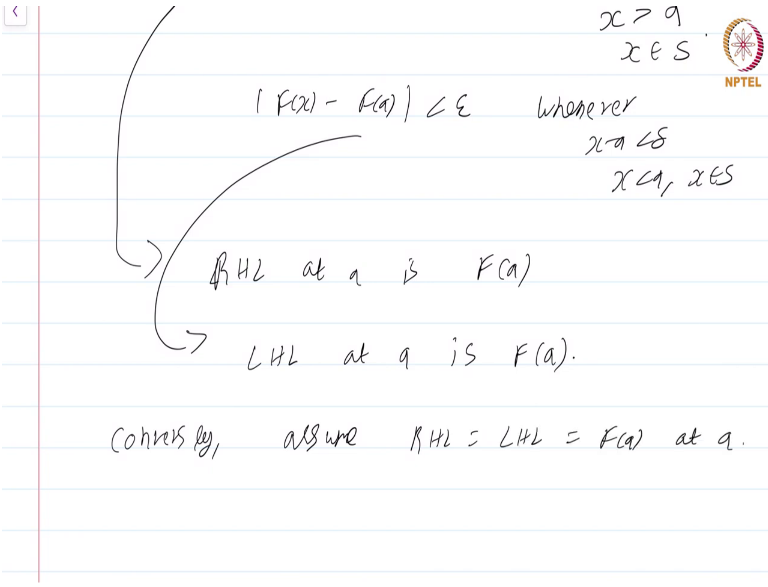

But this just says the right-hand limit at a is F(x) and this just says the left-hand limit at F(a) at a is F(a), right. These just follow from the very definition of limit excellent. So, we have shown that continuity means that the right-hand limit at a is F(a). Now, for the converse. Conversely assume right-hand limit equal to left-hand limit equal to F(a).

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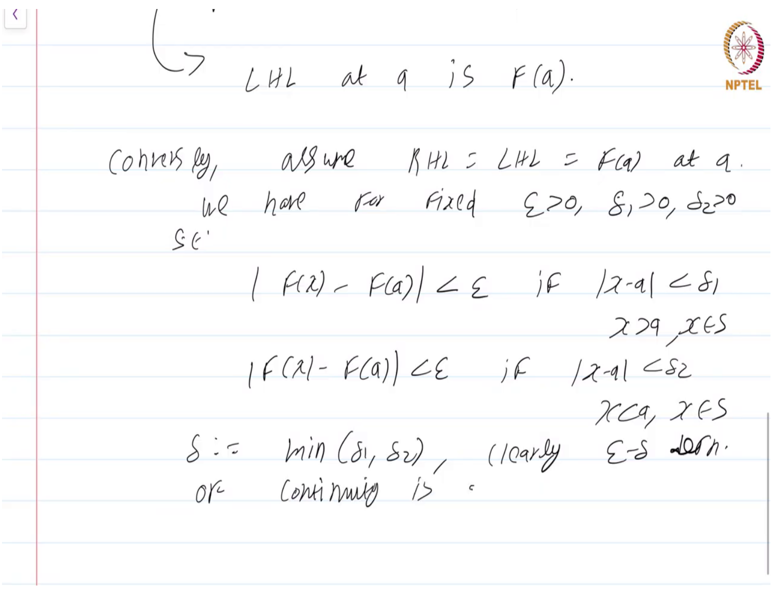

Then, what we have for fixed $\epsilon > 0$ $\delta_1 > 0,\ \delta_2 > 0$ such that $|F(x) - F(a)| < \epsilon$ if $|x - a| < \delta_1,\ x > a,\ x \in S$. Similarly, $|F(x) - F(a)| < \epsilon$ if $|x - a| < \delta_2,\ x < a,\ x \in S$.

I have just translated the left-hand limit existing at the point a and the right-hand limit existing at the point a into these conditions. So, if you choose $\delta := min\{\delta_1, \delta_2\}$. Clearly the $\epsilon - \delta$ definition of continuity satisfied.

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Conversely, assume RHL = LHL = F(a) at a
we have for fixed $\epsilon > 0$, $\delta_1 > 0$, $\delta_2 > 0$
s.t.

$$|F(x) - F(a)| < \epsilon \quad \text{if} \quad |x-a| < \delta_1, \; x > a, \; x \in S$$

$$|F(x) - F(a)| < \epsilon \quad \text{if} \quad |x-a| < \delta_2, \; x < a, \; x \in S$$

$\delta := \min(\delta_1, \delta_2)$, clearly

$$|F(x) - F(a)| < \epsilon \quad \text{if} \quad |x-a| < \delta, \; x \in S.$$

Therefore F is continuous at a.

So, better to not write it like this clearly $|F(x) - F(a)| < \epsilon$ if $|x - a| < \delta \quad x \in S$, this follows immediately. Therefore F is continuous at a. Now, if it turns out that only one of these limits is defined, let us say this limit is defined, then it is sufficient to show that this limit is equal to F(a).

Similarly, if only this limit exists, then it's sufficient to show that this limit I mean only if this limit is defined, it is sufficient to show that this limit is equal to F(a), then the function will be continuous. So, note that what we have shown is both the left-hand limit and the right-hand limit should both exist and should be equal to F(a) for continuity, it might be the case that F(x) left-hand limit is equal to the right-hand limit of F(x) at a, but they may not agree with the functional value.

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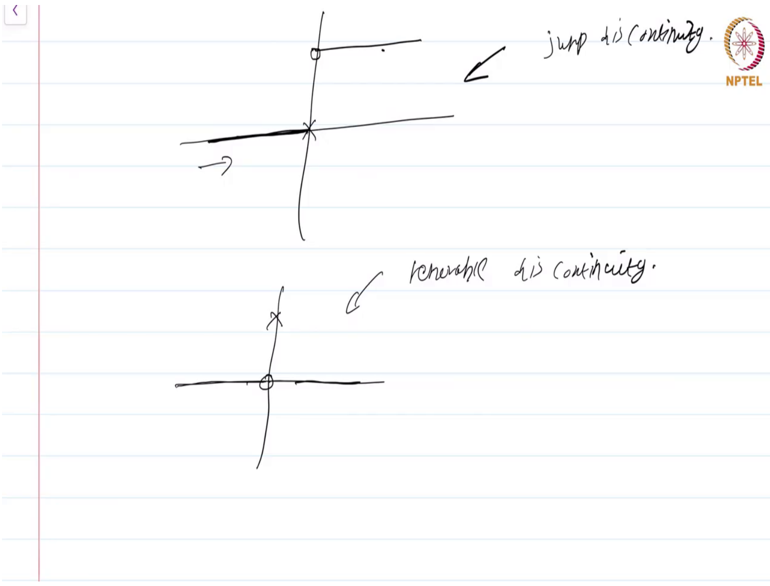

Let us see a few examples pictorially to see what and all can happen. So, take for instance, function which is 0 when $x \leq 0$, then 1 if x > 0. This side denotes by putting a circle here and a cross here to denote that at the point 0, the value is 0.

Now, here you note that the left-hand limit exists and is equal to the functional value, but the right-hand limit exists, but is not equal to the functional value. More interestingly, you consider this example 0 here, but 1 here, then both the left-hand limit and the right-hand limit will both exist but not be equal to the functional value at the point such a thing is sort of a removable discontinuity.

What I mean by that is you can make this function continuous by redefining the function at that one-point a, it is possible to make it continuous whereas, this function the above you has a jump discontinuity. It is not possible to redefine the function just at 0 to somehow make the function continuous that simply not possible.

So, you notice that studying left-hand limits and right-hand limits could be very much useful in studying the points of discontinuity and we will do this in great detail in a later module. Now, I want to end with one more remark which is essentially going to be an exercise.

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Remark: Suppose S is a set that has arbitrarily large elements.
Then observe that $F: S \to \mathbb{R}$

$$\lim_{x\to\infty} F(x) = \lim_{x\to 0^+} F\left(\frac{1}{x}\right)$$

where $F\left(\frac{1}{x}\right)$ is defined in an appropriate set with 0 as a limit point.
Similarly, we can deal with LHL.

Remark: Suppose S is a set that has arbitrarily large elements. Then observe that $\lim_{x\to\infty} F(x) = \lim_{x\to 0^+} F(\frac{1}{x})$ where F of 1 by x is defined in an appropriate set with 0 as a limit point.

Now, I am intentionally skipping the details. You can let me just write $x \to 0^+$ that is what relates the whole thing. You can view limit x going to infinity as right-hand limit of a function slightly different function defined using reciprocals, it is a function $F(\frac{1}{x})$ and this $\lim_{x\to 0^+} F(\frac{1}{x})$ you need to know where this function $F(\frac{1}{x})$ is defined that will be an appropriate set, I am going to leave the details to you.

So, the definition of infinite limits is actually subsumed by the definition of one-sided limit. One-sided limits sort of takes care of infinite limits also, we need not have defined infinite limits separately. Similarly, we can deal with left-hand limits fine. So, in the exercises, you will study this relationship in slightly more detail to understand what are the analogies between infinite limits and one-sided limits.

This is a course on real analysis, and you have just watched the module on one-sided limits.