Real Analysis - I
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Lecture – 16.2
Infinite Limits

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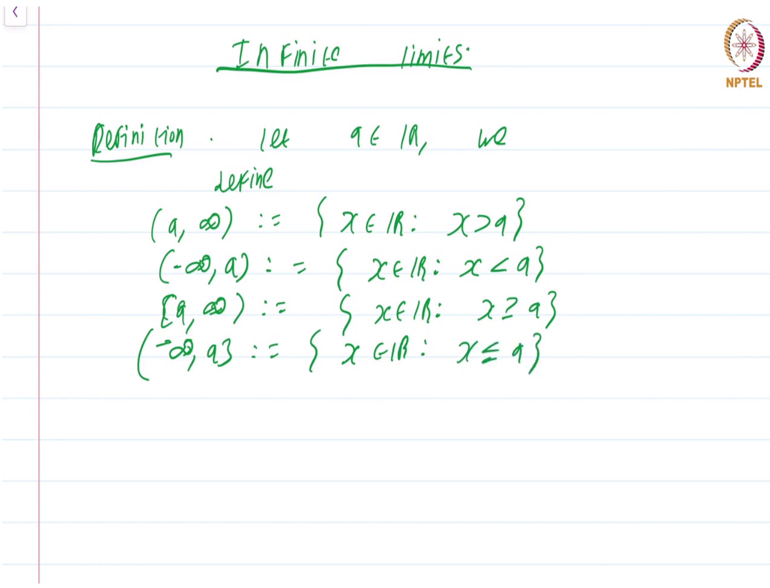

In this module, we are going to be talking about Infinite Limits. There are two types in fact, four types, but let us restrict ourselves to two types when x approaches infinity and when F(x) goes to infinity. So, let us make this precise. First, let me introduce some new types of intervals.

Definition. Let $a \in \mathbb{R}$. We define

$(a, \infty) := \{x \in \mathbb{R} : x > a\}$

$(-\infty, a) := \{x \in \mathbb{R} : x < a\}$

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$(-\infty, a) := \{x \in \mathbb{R} : x < a\}$

$(-\infty, \infty) := \mathbb{R}$

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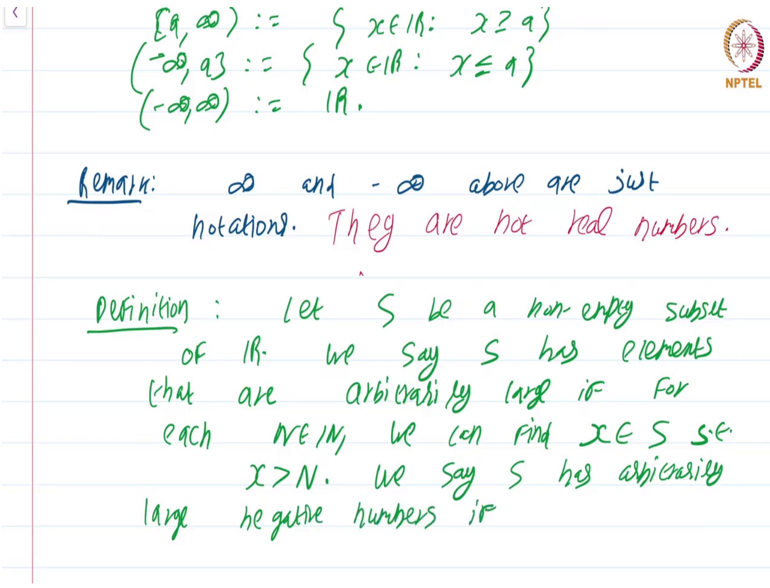

Now, we are just defining various intervals that have the symbols ∞ and $-\infty$. Remark: ∞ and $-\infty$ above are just notations. Let me put this in red, they are not real numbers.

In particular, it does not make sense in this course to talk about the set $[-\infty, a)$ for instance that simply does not make sense, they are just symbols. And, it does not make sense to ask what is $\infty - \infty$ or what is $\frac{\infty}{\infty}$ because we are not treating them as real numbers, they are just a convenient notation that will enable us to talk more freely about infinite limits.

So, there are several definitions we will have to make. So, first let us make the definition of a set having arbitrarily large elements.

Definition: let S be a non-empty subset of R. We say S has elements that are arbitrarily large if for each natural number $N \in \mathbb{N}$, we can find x in S such that x > N.

So, no matter what large natural number we choose. We can find some element x in the set S that is even larger. Similarly, we say S has arbitrarily large negative numbers if for each $N \in \mathbb{N}$ we can find $x \in S$ such that x < -N.

So, you can find elements that are as large as you desire, but negative. So, these are very nice formalizations of the intuitive notion of a set being arbitrarily large and arbitrarily small.

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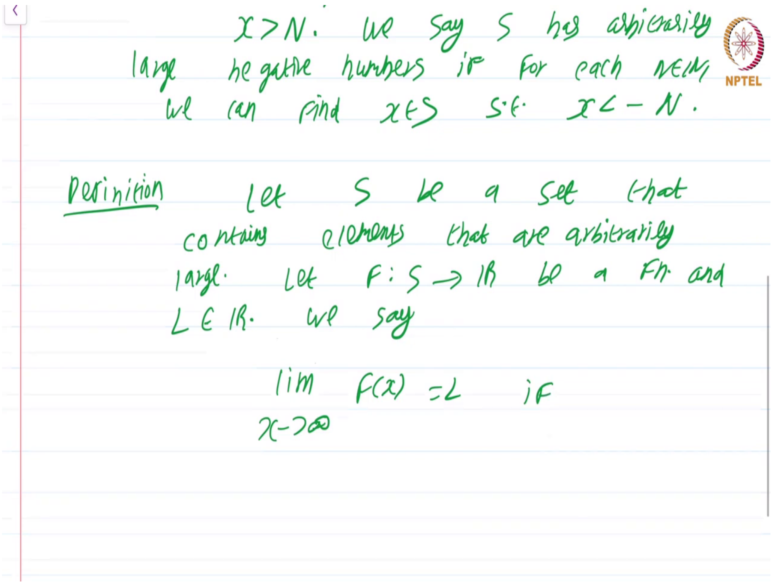

Now, we are going to try to define what it means for $\lim_{x \to \infty} F(x) = L$, that is the aim. So, definition so, let me just erase this

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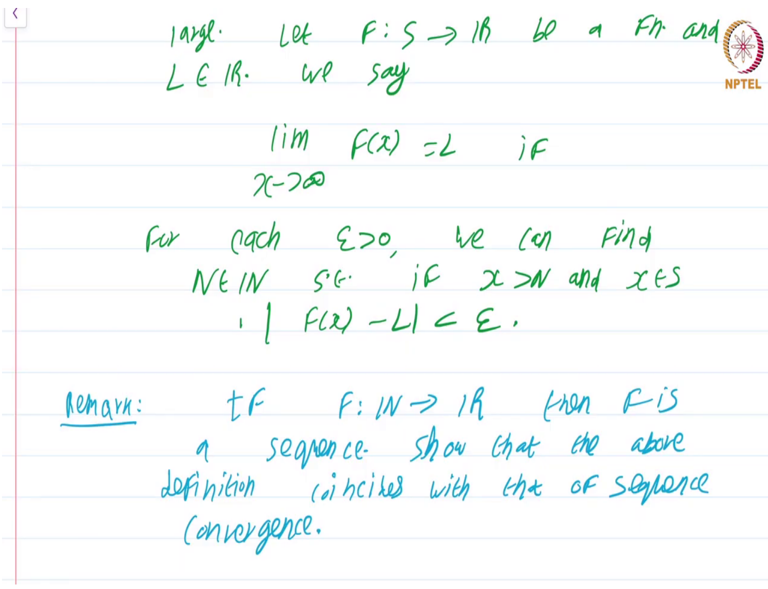

Definition: Let S be a set that contains elements that are arbitrarily large.

Let $F : S \longrightarrow \mathbb{R}$ be a function and $L \in \mathbb{R}$. We say $\lim_{x \to \infty} F(x) = L$ if for each $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if x > N and $x \in S$, $|F(x) - L| < \epsilon$.

The definition is a very parallel definition to that of finite limits. In the case of finite limits, you add x approaching a limit point. Here you have x approaching infinity. So, all you say is for each $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if x > N and $x \in S,\ |F(x) - L| < \epsilon$.

Remark: if $F : \mathbb{N} \longrightarrow \mathbb{R}$, then F is a sequence. Show that the above definition coincides with that of sequence convergence. So, this definition is certainly an extension of what we have already seen for sequences, but except now this set S could be a very arbitrary set not just natural numbers.

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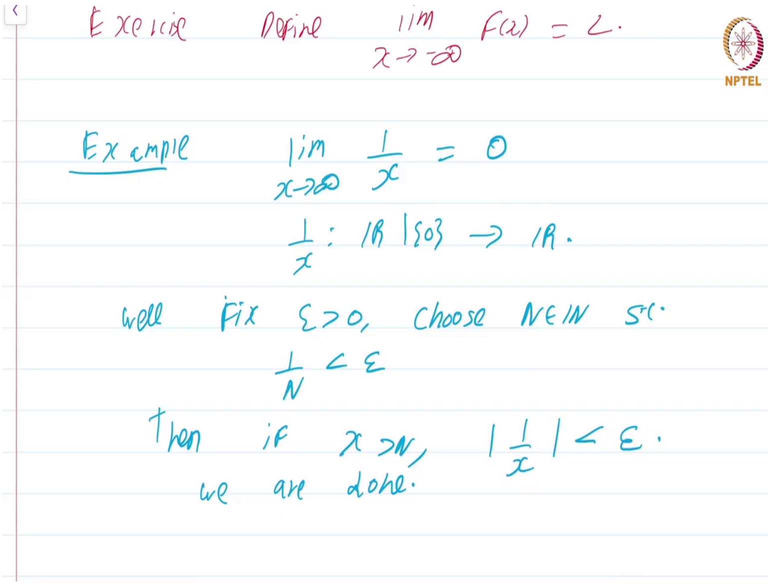

Exercise: Define $\lim_{x \to \infty} F(x) = L$.

Well, that is exactly the same analogous definition that you will have to make . So, this deals with what happens when x becomes large.

So,example $\lim_{x \to \infty} \frac{1}{x} = 0$. So, $\frac{1}{x}$ is, of course, defined on $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$. How do you solve this? Well, fix $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if x > N, $\frac{1}{x} < \epsilon$ and we are done. So, this was a fairly trivial proof. We have now shown that as $\lim_{x \to \infty} \frac{1}{x} = 0$.

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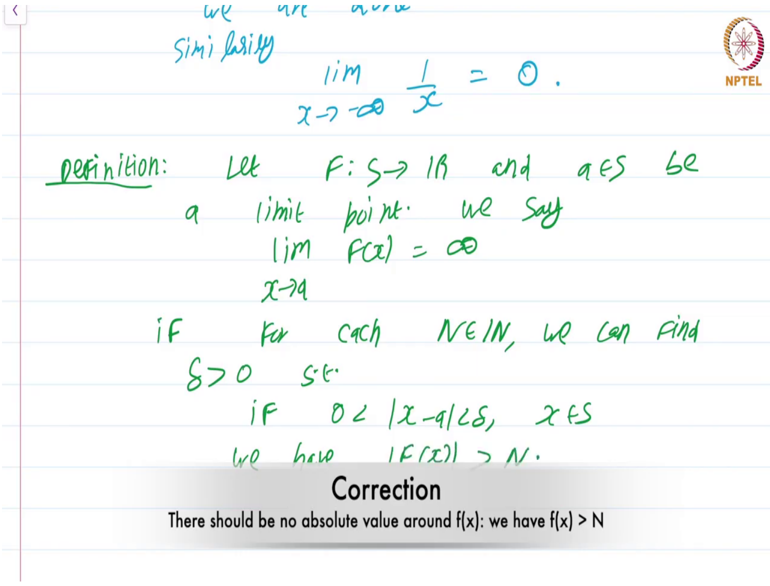

Similarly, $\lim_{x \to -\infty} \frac{1}{x}$ this also converges to 0. Then x going to $-\infty$ does not change the behavior $\frac{1}{x}$ still goes to 0 not to -0 which makes no sense. Now, we come to the second definition.

Let $F : S \longrightarrow \mathbb{R}$ and $a \in S$ be a limit point. We say $lim_{x \to a} F(x) = \infty$ if for each $N \in \mathbb{N}$, we can find $\delta > 0$ such that if $0 < |x - a| < \delta$ and $x \in S$, we have $|F(x)| > N$.

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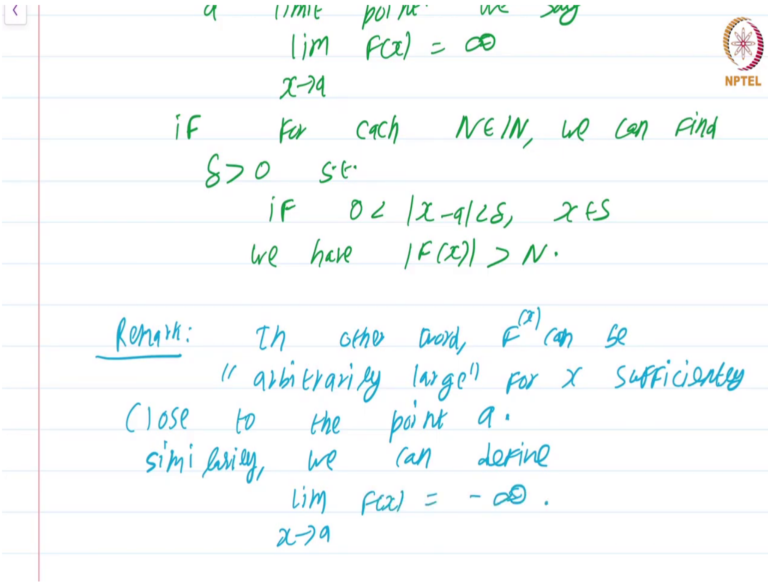

In other words, let us make a remark, this definition is fairly easy, but this remark is nice and expressive. In other words, F can be made arbitrarily large for x sufficiently close to the point a. So, I am not even going to bother trying to define precisely what arbitrarily large is from the way this definition is phase, it should be clear to you what it means.

So, we have now defined what it means for $lim_{x \to a} F(x) = \infty$. Similarly, we can define $lim_{x \to a} F(x) = -\infty$, that definition is exactly analogous.

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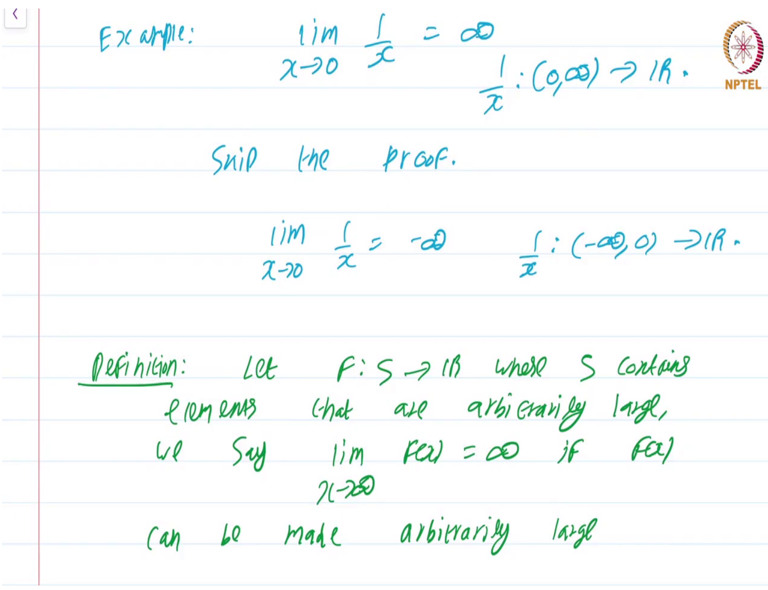

So, again the example $\lim_{x \to 0} \frac{1}{x} = \infty$. Now, is this correct first of all? Let us think about it. What happens when you approach x from the left-hand side? Well, x blows up, $\frac{1}{x}$ blows up, but $\frac{1}{x}$ blows up negatively. So, here I have to specify $\frac{1}{x}$ is defined from $(0, \infty)$ to $\mathbb{R}$. I can get away with this by renaming I mean redefining $\frac{1}{x}$ to be only on the positive numbers.

Now, this is fairly easy to show. So, I am going to skip the proof and leave you to do it. This is fairly straightforward . Similarly, $\lim_{x \to 0} \frac{1}{x} = -\infty$, then $\frac{1}{x}$ is defined from $(-\infty, 0)$ to $\mathbb{R}$ that that proof is also entirely similar.

So, final definition and this I am going to use the full force of language.

Let $F : S \longrightarrow \mathbb{R}$ where S contains elements that are arbitrarily large. We say $\lim_{x \to \infty} F(x) = \infty$, if F(x) can be made arbitrarily large by making x arbitrarily large.

So, I am using the full force of the language that we have developed to define this. I leave it to you to make this mathematically precise by getting rid of these words arbitrarily large and so on. Actually, it really does not matter if you understand what it means mathematically that is more than sufficient. So, now, we have extended the notion of limits to involve various infinite quantities. Similarly, I leave it to you to do it for $-\infty$ and various combinations that may have been left out.

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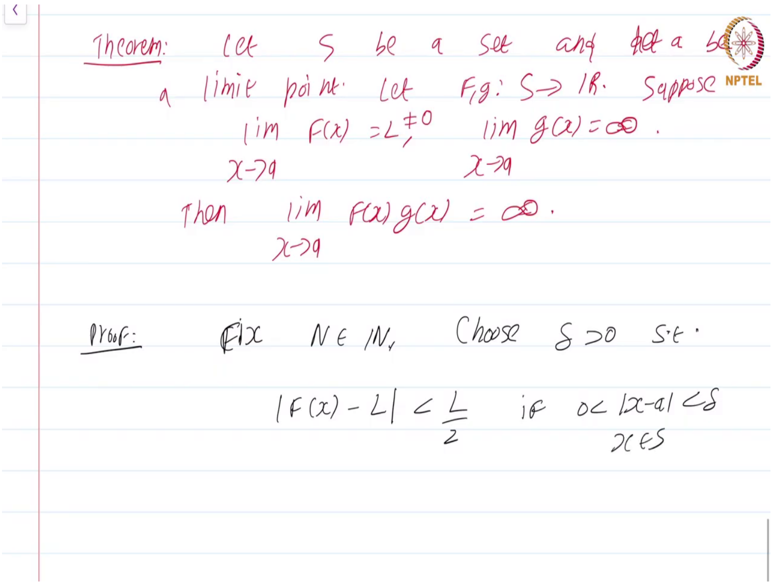

Let me just prove one theorem. There are various limit laws that we have shown are right and many of them do generalize when we have infinite limits, you can look through the exercises and solve them to see which of these go through. I am just going to prove one of them.

Let S be a set and let 'a' be a limit point. Let $F, g : S \longrightarrow \mathbb{R}$. Suppose $\lim_{x \to a} F(x) = L$ and $\lim_{x \to a} g(x) = \infty$. Then, $\lim_{x \to a} F(x)g(x) = \infty$.

So, if you have one limit going to infinity and the other limit going to a finite quantity, then the product limit goes to plus infinity. Now, you should be a bit careful this result is valid, but what will happen if F(x) goes to 0? Right. So, L is not 0, we have to assume that, $L \neq 0$. If L = 0, you cannot really predict what is happening. So, look through the exercises for careful statements of other limits laws.

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Proof: Fix $N \in \mathbb{N}$, Choose $\delta > 0$ s.t.

$|F(x) - L| < \frac{L}{2}$ if $0 < |x-a| < \delta$, $x \in S$

$|g(x)| > \frac{2N}{L}$ if

$|F(x)| > \frac{L}{2}$

$|F(x) g(x)| > N$ if $0 < |x-a| < \delta$, $x \in S$

Hence proved.

So, proof this particular limit law does require $L \neq 0$.

Proof: Choose so, fix we have to ultimately show that product is ∞. So, fix $N \in \mathbb{N}$. Now, choose $\delta > 0$ such that first of all $|F(x) - L| < \frac{L}{2}$ if $0 < |x - a| < \delta$ and $x \in S$ first of all and second $|g(x)| > \frac{2N}{L}$, if 0 less than whatever the same thing.

So, choose δ such that both these conditions happen. This can be done because you can choose δ such that the first condition is true, you can choose δ such that the second condition is true therefore, you can choose a δ that is the minimum of both such that both conditions are satisfied.

Now, the first condition immediately gives that $|F(x)| > \frac{L}{2}$. So, again $L \neq 0$ and L> 0 that is a better condition F(x) = L > 0 that is required. Excellent, we have both these.

So, because $|F(x)| < \frac{L}{2}$ and $|g(x)| > \frac{2N}{L}$, $|F(x)g(x)|$ is simply greater than N if $0 < |x - a| < \delta$ and $x \in S$. This is an immediate consequence hence proved.

So, if you have one quantity converging to a positive number and another quantity converging to infinity, then the product will have to also converge to infinity. So, this concludes this module, please do look through the exercises for many other limit laws involving infinite quantity.

This is a course on Real Analysis and you have just watched the module on Infinite Limits.