Real Analysis - I
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Lecture – 15.5
Global Continuity and Open Sets

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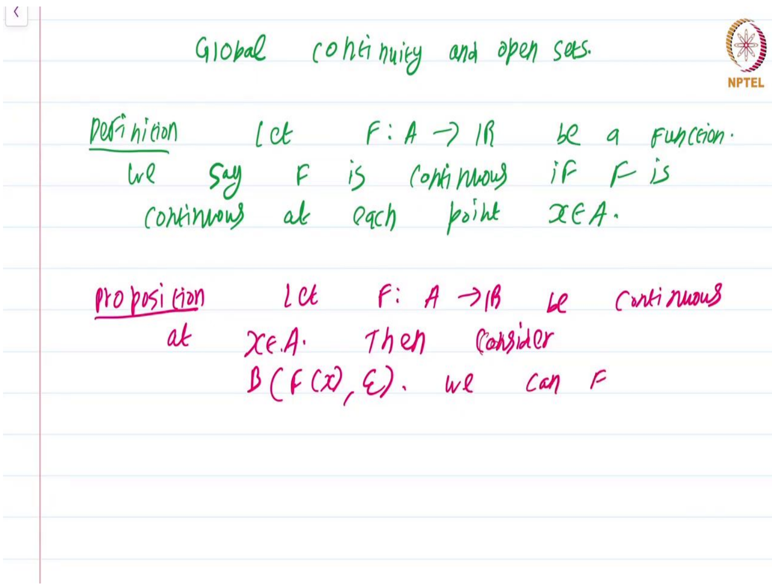

We begin with a definition.

Definition: Let $F : A \longrightarrow \mathbb{R}$ be a function. We say F is continuous if F is continuous at each point x in A.

So, a function that is continuous at each point of the set is known as a continuous function. Now what we want to do is, we want to consider functions that are globally continuous; that means, throughout their domain A; at every point in the domain they are continuous and I want to see whether I can characterize this in some way.

So, to do that first let me state a proposition.

Proposition: Let $F : A \longrightarrow \mathbb{R}$ be continuous at x in A; at just one point. Then consider $B(F(x), \epsilon)$. So, consider a ball of radius epsilon around the point F(x), we can find or rather let me instead of writing consider.

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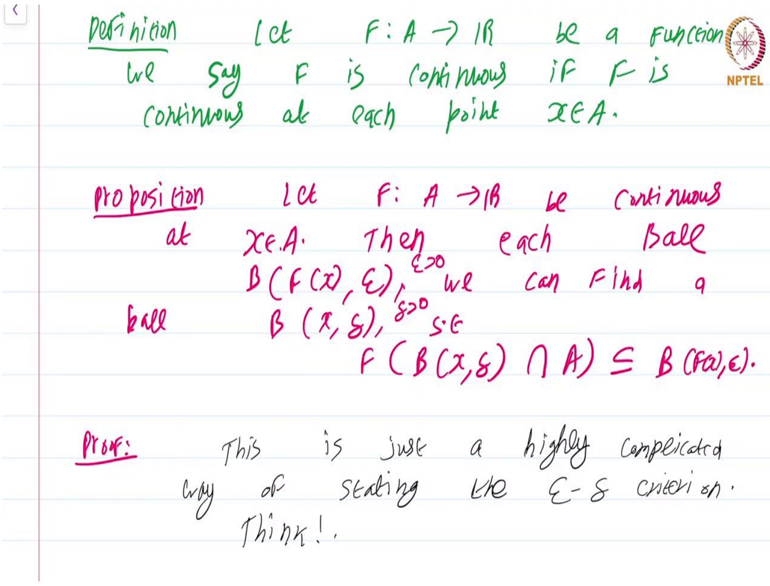

Then for each ball $B(F(x), \epsilon)$ we can find a ball $B(x, \delta)$ such that $F(B(x, \delta) \cap A) \subset B(F(x), \epsilon)$. Look at the point F(x), consider a ball $B(F(x), \epsilon)$, we can find a ball $B(x, \delta)$. Of course, $\delta > 0$; such that $F(B(x, \delta) \cap A)$ is contained in $B(F(x), \epsilon)$; again here also let me just mention $\epsilon > 0$.

Proof: This is just a highly complicated way of stating the $\epsilon - \delta$ criterion right, think! So, this fancy proposition just recasts the $\epsilon - \delta$ definition in the language of balls. Now why is this recast into the language of balls useful? Because we can prove the following theorem.

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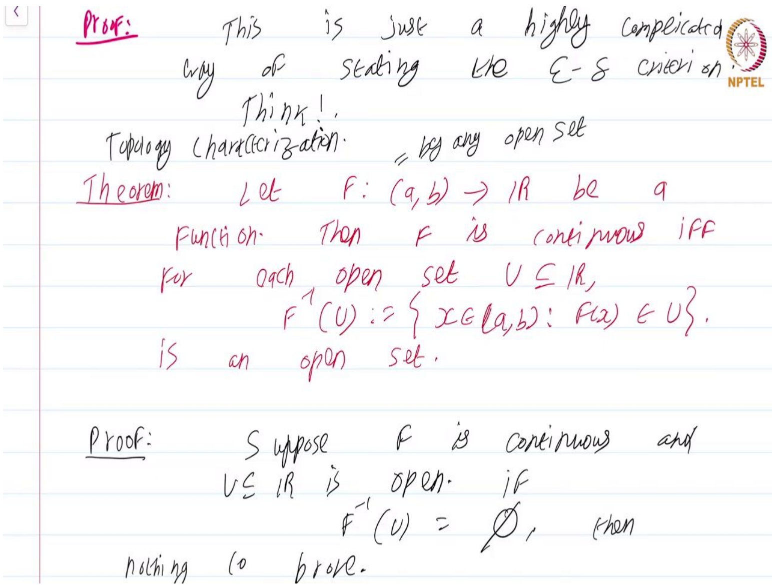

The importance of this theorem will not be clear to you until you do a course in topology, but since this is so important I am stating it.

Theorem: Let $F:(a,b)\longrightarrow\mathbb{R}$ be a function. Then F is continuous if and only if for each open set $U\subset\mathbb{R}$, the inverse image $F^{-1}(U)=\{x\in(a,b):f(x)\in U\}$; this is the pre image or the inverse image.

Please again do not confuse this with the inverse of a function, this is going to be a set; $F^{-1}(U)$ is an open set. Let us prove this theorem.

Proof: Suppose F is continuous and U subset of R is open. If F inverse U is the empty set this is a reasonable possibility the function F need not attain any value in the set U that is very much possible.

If $F^{-1}(U)$ is the empty set then nothing to prove. Why? Because the empty set for extraordinarily stupid reasons is an open set the condition of open set is every point is an interior point there is no point in the empty set. Therefore, it is an open set for stupid reasons; there is nothing to prove.

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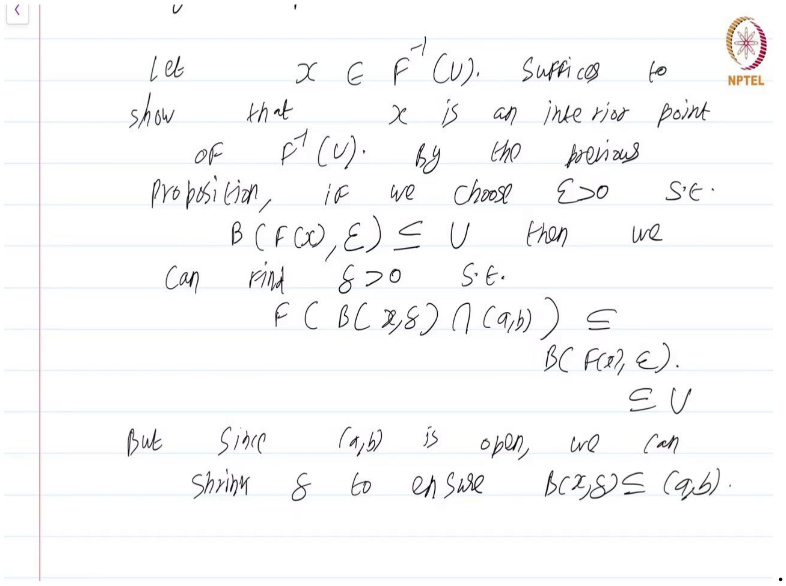

Let $x \in F^{-1}(U)$; we are assuming now that $F^{-1}(U)$ is a non empty set. Suffices to show that x is an interior point of $F^{-1}(U)$, right that is the definition of an open set, every point is an interior point.

Now, what we do is by the previous proposition. If we choose $\epsilon > 0$, such that $B(F(x), \epsilon)$ is contained in U, which we can do because F(x) is an element of U and U is an open set; then, we can find $\delta > 0$ such that $F(B(x, \delta) \cap (a, b)) \subset B(F(x), \epsilon) \subset U$.

This is exactly what the previous proposition said $F(B(x, \delta) \cap A) \subset B(F(x), \epsilon)$. Note that this is contained in U that is how we chose our ϵ. We chose our ϵ in such a manner that $B(F(x), \epsilon)$ is contained in U. But since (a, b) itself is open we can shrink δ to ensure $B(x, \delta)$ is contained in the open set (a b), this can be done.

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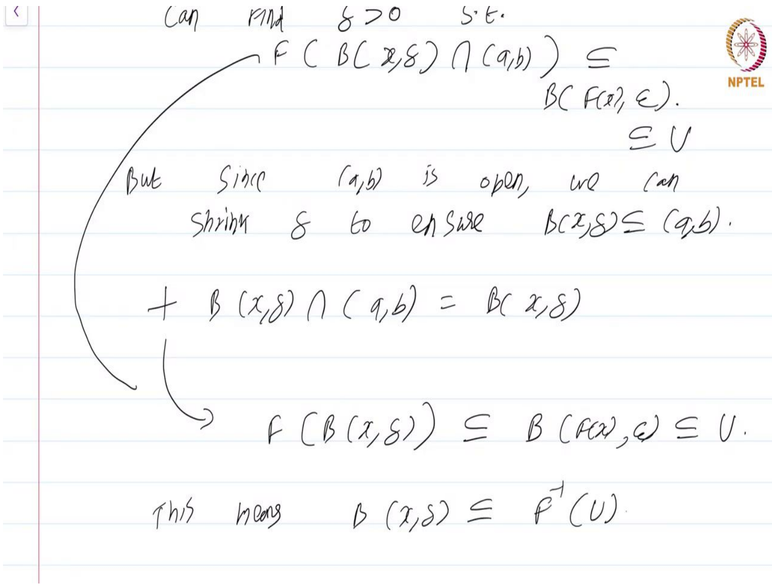

Once you do that what we get is $B(x, \delta)$ intersect (a,b); well, no shock here it is just $B(x, \delta)$ because we have shrunk δ to assume that $B(x, \delta)$ is contained in (a,b). Now, immediately this plus this gives $F(B(x, \delta)$ is contained in $B(F(x), \epsilon)$ is contained in U right, but this means $B(x, \delta)$ is a subset of $F^{-1}(U)$.

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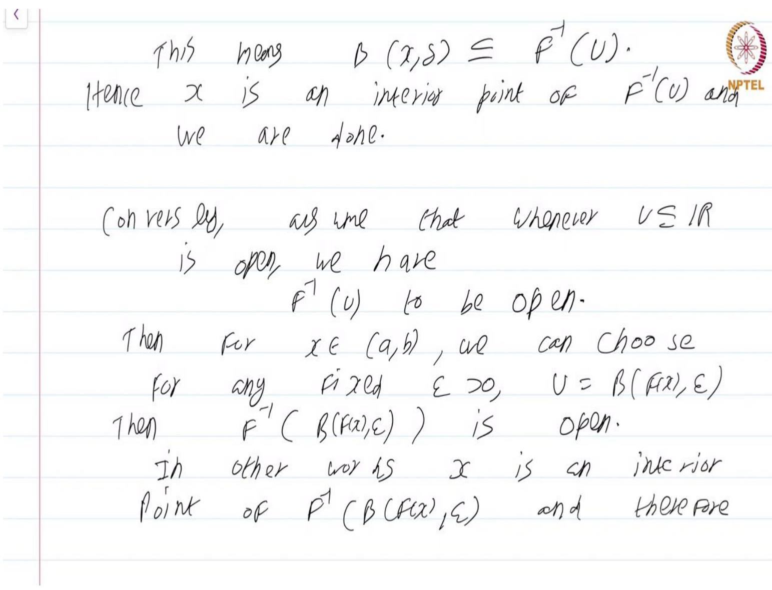

In other words x is an interior point, hence x is an interior point of $F^{-1}(U)$ and we are done. So, this proves one direction assuming F is continuous. We got this.

Now conversely assume that whenever $U \subset \mathbb{R}$ is open, we have $F^{-1}(U)$ to be open. Then, for x in (a,b), we can choose for any fixed $\epsilon > 0$, U has $B(x, \epsilon)$ right; because this applies to any open set certainly applies to open balls.

Then, $F^{-1}(B(F(x), \epsilon))$ is open. In other words x is an interior point of $F^{-1}(B(F(x), \epsilon))$.

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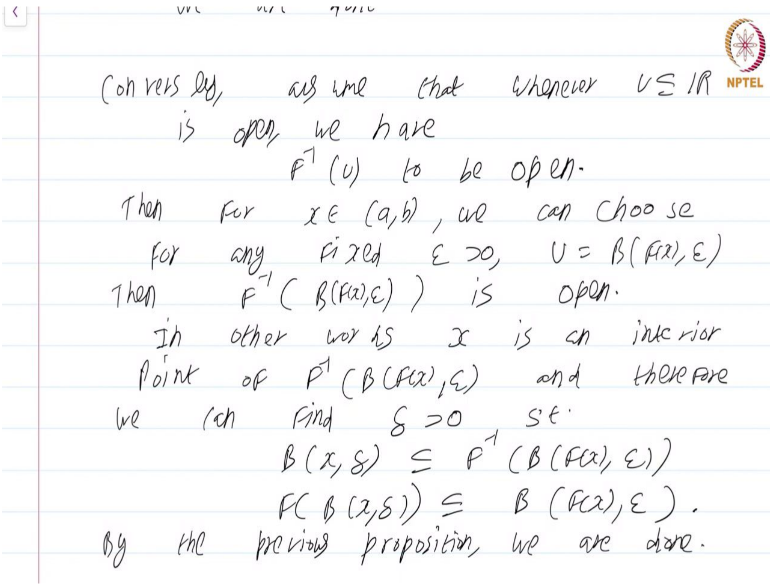

Therefore, we can find $\delta > 0$ such that $B(x, \delta)$ is contained in $F^{-1}(B(F(x), \epsilon))$ which is just saying $F(B(x, \delta))$ is contained in $B(F(x), \epsilon)$, that by the previous proposition we are done.

So, what does this say? This theorem says that if you are starting with a function from an open interval then the function F is continuous if and only if it pulls back open sets in R to open sets.

Now, if you think about this for a moment, if you look through the proof carefully this (a,b) could have been replaced by any open set and this and the same proof will go through with minor modifications. Please think about that. So, let me just mention that this is called the topological characterization of continuity .

This is a course on Real Analysis and you have just watched the module on Global Continuity and Open Sets.