Real Analysis - I

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Lecture – 15.3
Limit of Sin X_X

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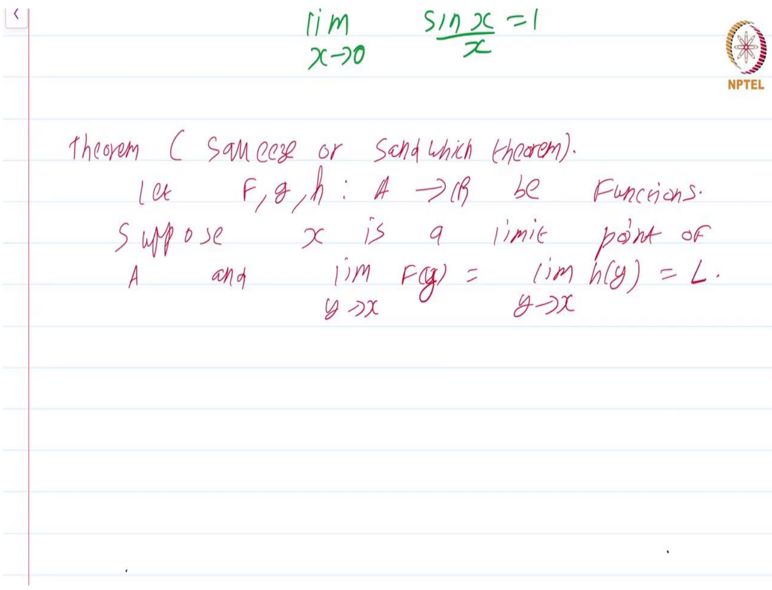

We come to what is arguably, one of the most famous limits that you have learned so far, $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Now, to prove this I would require some elementary properties of the trigonometric functions. Now, these can be proved directly by understanding that sine and cosine and so on can be defined using circles and triangles within circles and using theorems from geometry.

I do not want to do that. I want to give a purely analytic proof by which I mean sine, cosine, etc. I want them to be defined using analysis and their basic properties also proved using analysis. Now, I would define these trigonometric functions at a later point of time, meanwhile you will have to take some basic facts about trigonometric functions on faith.

Of course, you have manipulated these identities quite a bit in your high school. Just take them for granted for the time being.

Now, to prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, I am not going to do it directly, I am just going to invoke this very useful theorem which we have already seen for sequences.

Theorem (Squeeze or Sandwich theorem) Let $F, g, h : A \longrightarrow \mathbb{R}$ be functions. Suppose, x is a limit point of A and $\lim_{y \to x} F(y) = \lim_{y \to x} h(y) = L$.

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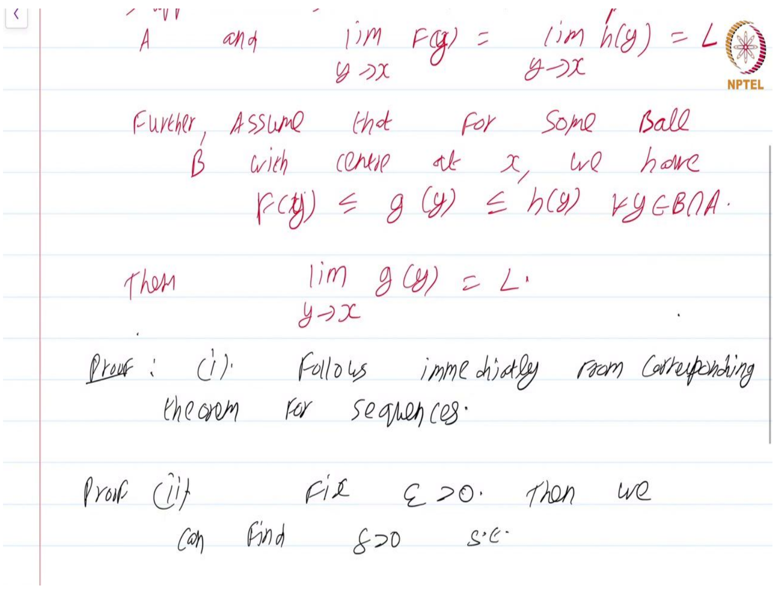

Further assume that for some ball B with centre at x, we have $F(y) \leq g(y) \leq h(y) \; \forall y \in B \cap A$. Then $\lim_{y \to x} g(y) = L$.

Proof: First proof follows immediately from the corresponding theorem for sequences. Let us give another proof directly using $\epsilon - \delta$.

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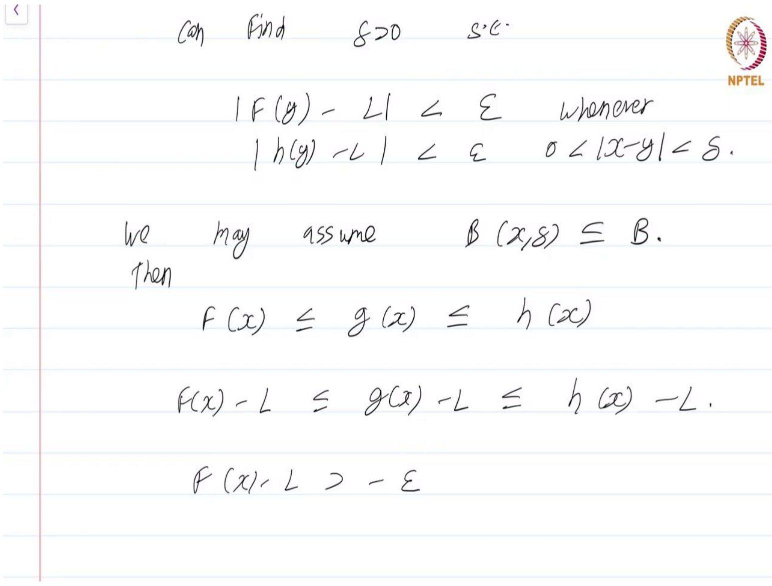

Proof 2- Fix $\epsilon > 0$, then we can find $\delta > 0$, such that $|F(y) - L| < \epsilon$ and simultaneously $|h(y) - L| < \epsilon$, whenever $|x - y| < \delta$. This just comes directly from the $\epsilon - \delta$ definition applied to F and h then taking the minimum of the two δ's.

Since I have done such things quite a bit for sequences, I will not be explicitly saying anymore how exactly this δ was constructed satisfying these two conditions simultaneously. Now, we may assume $B(x, \delta)$ is contained in B. Why can we assume this? If not just make δ even smaller in order to make $B(x, \delta)$ contained in B, this can be done. Then we know that $F(x) \leq g(x) \leq h(x)$.

And now the proof should be very very similar to what we did for sequences. We get $F(y) - L \leq g(y) - L \leq h(y) - L$. This just follows by subtracting L throughout the equation.

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we may assume $B(x,\delta) \subseteq B$.

Then

$$F(y) \leq g(y) \leq h(y)$$

$$F(y) - L \leq g(y) - L \leq h(y) - L.$$

$$F(y) - L > -\varepsilon$$

$$h(y) - L < \varepsilon.$$

$$-\varepsilon \leq g(y) - L \leq \varepsilon$$

and we are done.

But, $F(y) - L > -\epsilon$, whereas, $h(y) - L < \epsilon$. Why does this just follow from the fact that $|F(y) - L| < \epsilon$ and $|h(y) - L|$ is also less than ϵ right? That immediately gives $-\epsilon \leq g(y) - L \leq \epsilon$ and we are done. I will not belabor the proof anymore.

So, this shows that the squeeze theorem is true for functional limits also and if you notice this proof is more or less the same as the squeeze theorem for sequences. So, at this point let me make a parenthetical remark that is sort of beyond this course.

Now, you might think why do we waste time first doing sequences and proving some theorems when doing the exact same theorems for functional limits once more and later in the course you will see something called uniform convergence and the same things you will do for uniform convergence also.

You have various notions of convergence and you have similar theorems for all of them and you are wondering, why is it that we are not unifying all of them. They can be unified.

You can give a general notion of limit using some abstract machinery. This has been done by the mathematician A. F. Beardon. Just Google A. F. Beardon limits and you will get this. This is just some extraneous things. If you are interested you can pursue this.

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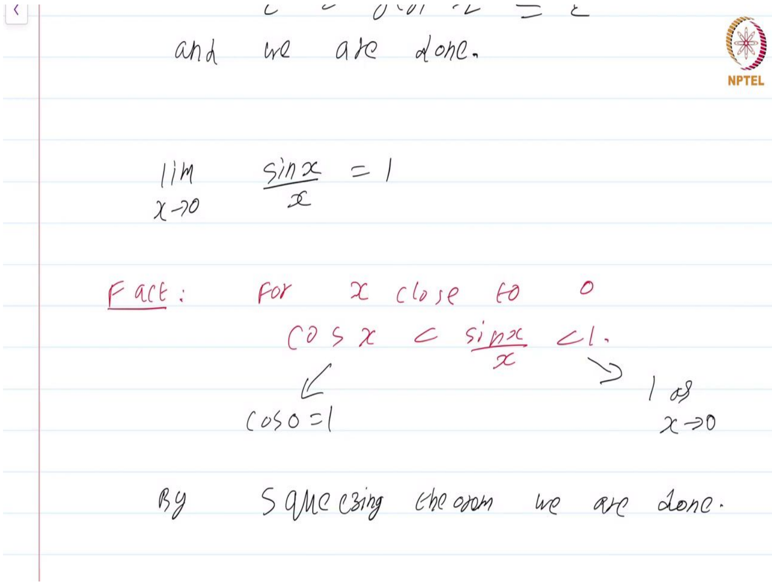

Coming back to more pressing issues, how do we use this to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$, how do we do this? Well, I need to use a fact.

Fact: For x close to 0, $\cos x < \frac{\sin x}{x} < 1$. This will be proved at a later point when I define the trigonometric functions precisely.

Now, because of this notice that the right hand side is a constant function. This converges to 1 as x goes to 0 and this cos x is a continuous function that is another fact that you know and cos 0 is 1.

Therefore, by squeezing theorem we are done. So, this is a short proof. Of course, all the heavy lifting has been shoved under the carpet and left for a later time, but once we do that this is fairly straightforward.

The squeeze theorem is very very useful. Again just as in sequences directly using the $\epsilon - \delta$ definition to show that some limits are true is a very very bad approach. You must use it as a last resort, when every other thing fails.

First try to apply the various limit laws, squeeze theorem and so on and try to get the limits. If everything fails, resort to using $\epsilon - \delta$.

This is a course on real analysis and you have just watched the proof that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.