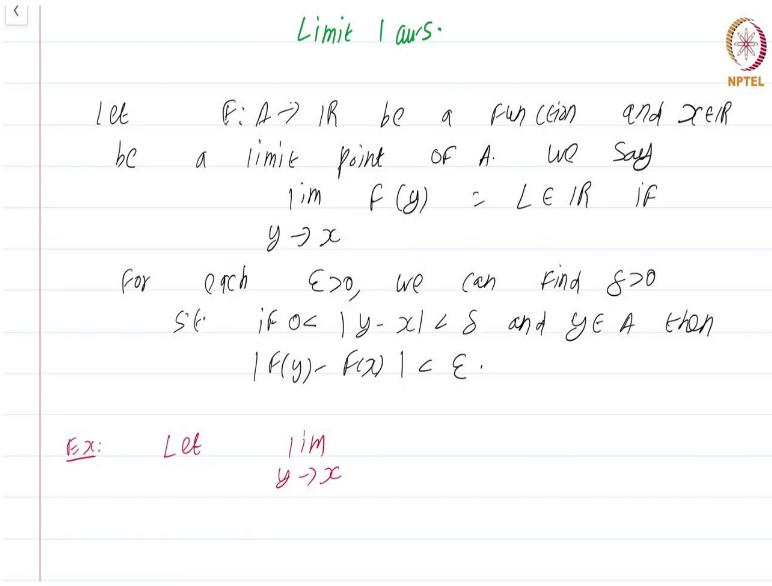
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 15.2
Limit Laws

(Refer Slide Time: 00:13)

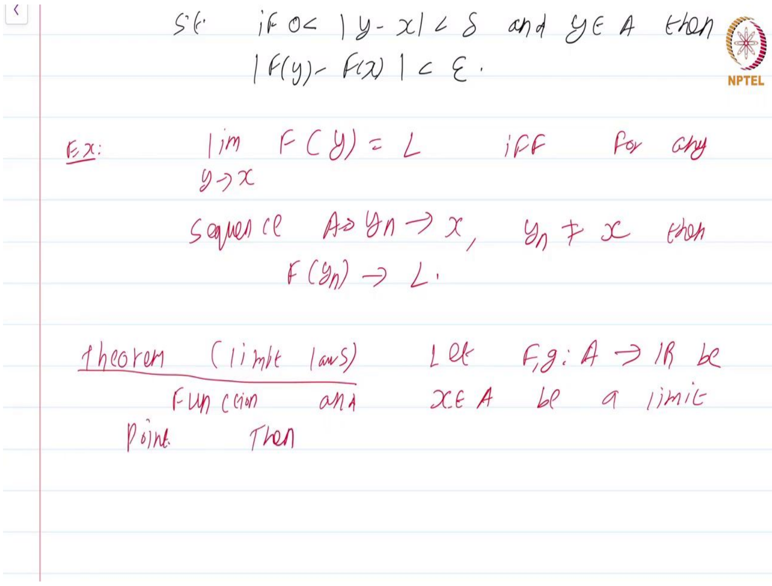


Let me recall the definition of a limit.

Let $F : A \longrightarrow \mathbb{R}$ be a function and $x \in \mathbb{R}$ be a limit point of A. We say $\lim_{y \to x} F(y) = L \in \mathbb{R}$ if for each $\epsilon > 0$, we can find $\delta > 0$, such that if $0 < |y - x| < \delta$ and $y \in A$, then $|F(y) - F(x)| < \epsilon$.

I am going to leave you with an exercise which is fairly easy; because we have done the heavy lifting, when we characterized the three possible definitions of continuity. Let limit y going to x or rather let me state it in the different way.

(Refer Slide Time: 02:18)



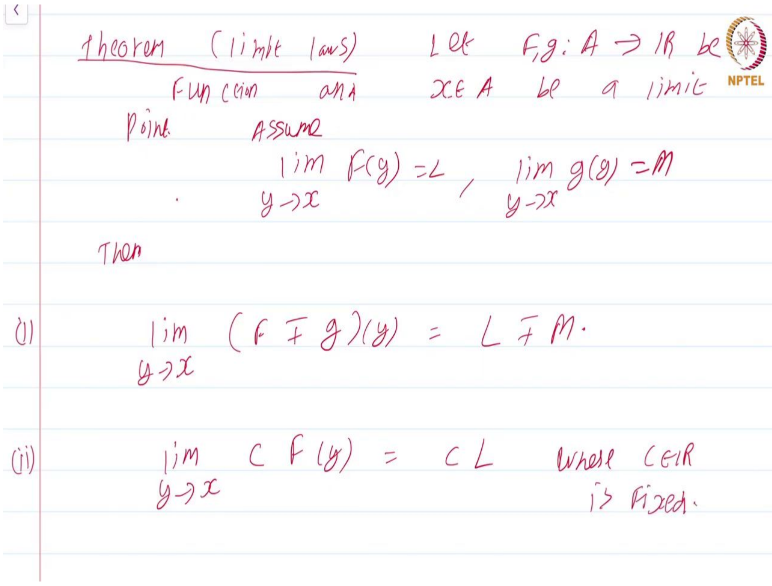
$\lim_{y \to x} F(y) = L$ if and only if for any sequence $y_n \in A$ converging to x and y_n is never x, then $F(y_n) \to L$. Now, again one of the reasons why we took x to be a limit point and not just any adherent point is because; there might not be any sequence satisfying the condition y_n converging to x and $y_n \neq x$, unless this point x is a limit point and not merely any old adherent point.

So, what this says is, you can characterize limits of functions entirely using the notion of limits of sequences. And the proof of this involves a bit of work; but we have already done that work when we saw that the three definitions of continuity are equivalent, all you have to do is go through that proof thoroughly and write down a proof of this. So, this is my strategy of forcing you to re watch that lecture once more.

Now, that this is done; we save a lot of effort in the next theorem on limit laws.

Theorem (limit laws), let $f, g : A \longrightarrow \mathbb{R}$ be functions and $x \in A$ be a limit point; same setup except now we have two functions.

(Refer Slide Time: 04:45)

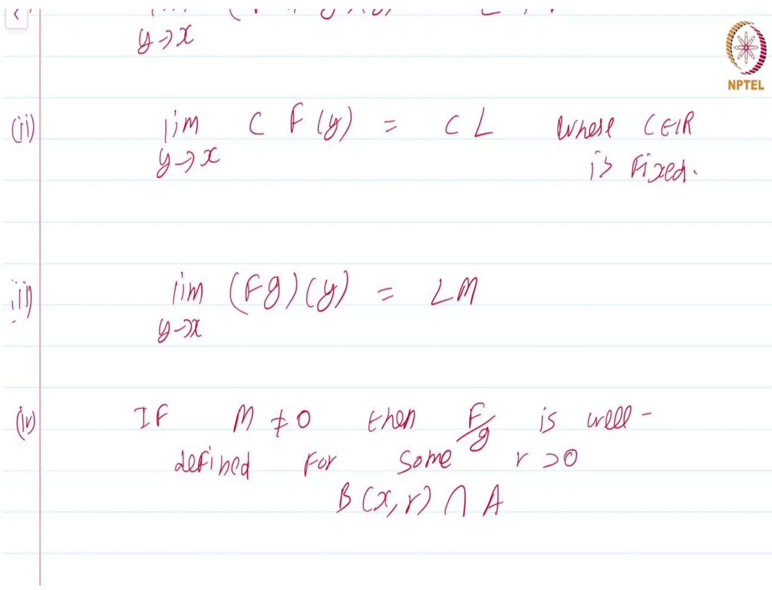


Assume $\lim_{y \to x} f(y) = L$ and $\lim_{y \to x} g(y) = M$ then

(i) $\lim_{y \to x}(f \pm g)(y) = L \pm M$

(ii) $\lim_{y \to x} cf(y) = cL$, where $c \in \mathbb{R}$ is fixed.

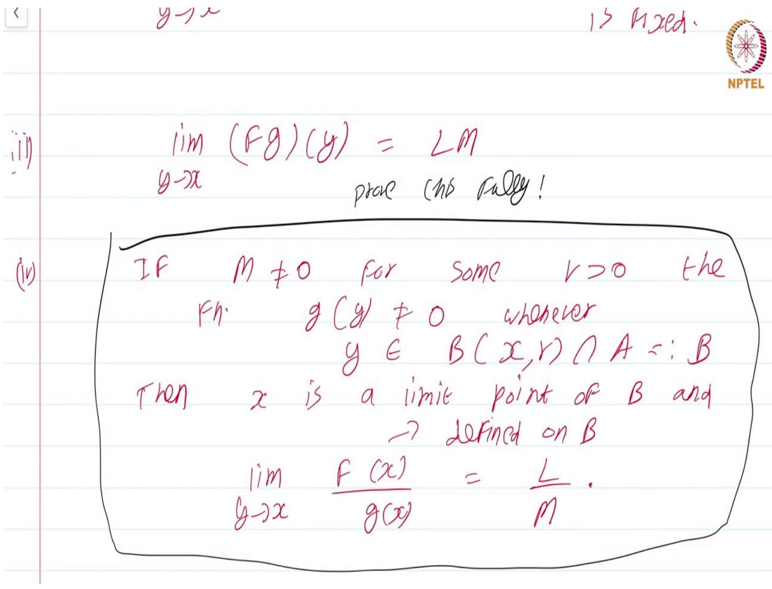
(Refer Slide Time: 05:46)



(iii) $\lim_{y \to x}(fg)(y) = LM$.

(iv) If $M \neq 0$, then f/g is well defined for some B(x,r) intersect A. If M is not 0, that is the $\lim_{y \to x} g(y) \neq 0$; then f/g is actually well defined for some r > 0. So, let me just write this precisely.

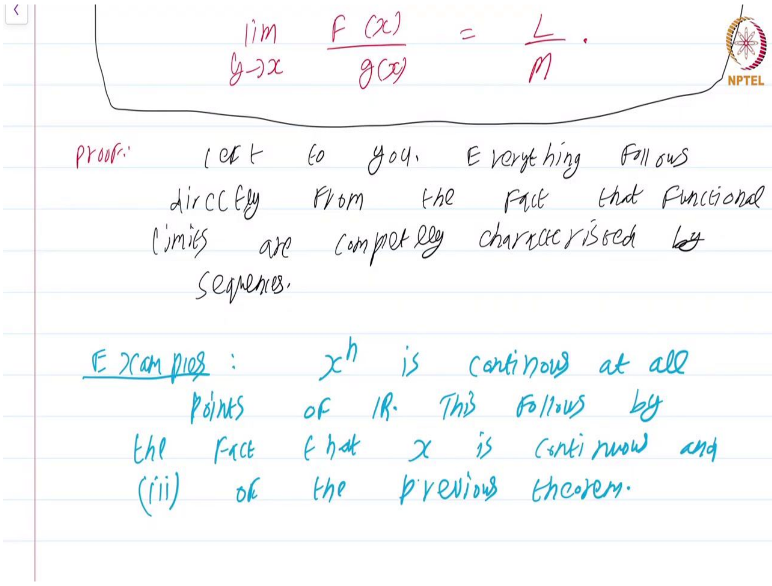
(Refer Slide Time: 07:07)



Then for some r > 0, the function g(y) is not 0 whenever y belongs to $B(x,r) \cap A$, which I defined to be the set B. Then x is a limit point of B and $\lim_{y \to x} \frac{f(x)}{g(x)}$, this is defined on B this is defined on B is equal to $\frac{L}{M}$.

A bit wordy, but I want it to be a bit precise; all I want to say is, if the limit of the denominator is not 0 then when you are sufficiently close to the limit point x, the function itself is not 0. Therefore, you can define the quotient f/g on some ball B(x,r) intersect A, which I am calling the set B. So, I am taking the limit of this function defined on B at the point x; to do this x must be a limit point that is part of the statement and that limit is equal to $\frac{L}{M}$.

(Refer Slide Time: 08:55)

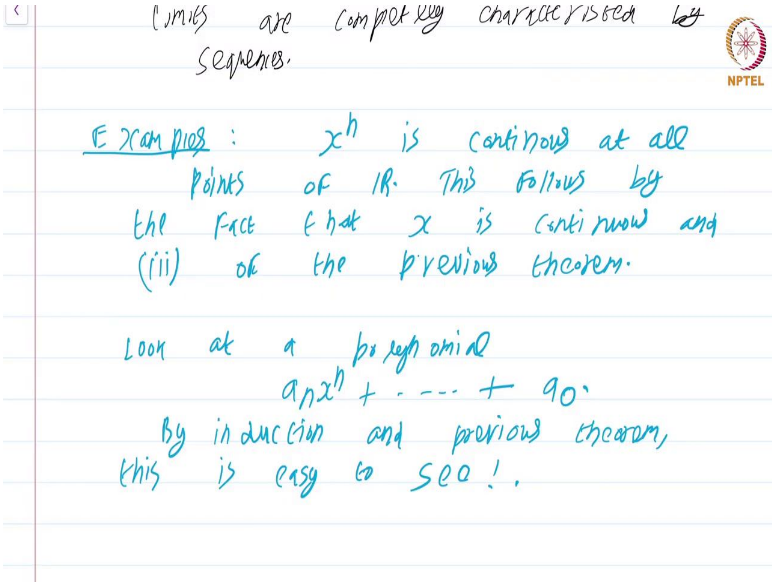


Now, proof, left to you, this might sound a bit harsh, I am leaving everything for you, but everything follows directly from the fact that functional limits are completely characterized by sequences, completely characterized by sequences. All of these proofs follow immediately by applying the previous exercise which again I have left for you as an exercise, everything follows immediately from this and the corresponding theorem for sequences.

Now, I recommend that you do not waste your time writing out this entire proof in full detail, rather write a complete proof of the last part; write a complete proof by yourself, prove this fully. If you can do this last part, the other three will be a piece of cake. So, the limit laws follow immediately from the fact that we have already done the work for sequences. Now, immediately we get nice consequences, we get nice consequences.

Examples, x^n is continuous at all points of R, this follows by the fact that x is continuous and (iii) of the previous theorem. We have already shown that the function x is continuous. Now, you have to show that x^n is continuous; the previous theorem will do the job for us.

(Refer Slide Time: 11:50)



Excellent, yet another thing; look at a polynomial which recalls an expression that looks like this; $a_n x^n + \dots + a_0$. Now, again by induction and previous theorem this is easy to see. So, we immediately got a good collection of examples of continuous functions just by applying the previous theorem.

This is a course on real analysis and you have just watched the module on limit laws.