Real Analysis - I

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Lecture - 2.1
Basic Set Theory

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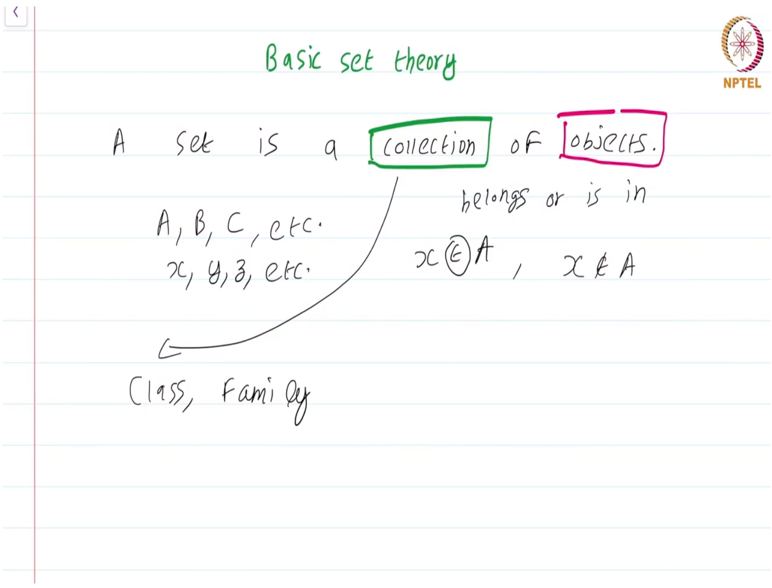

In this module, we will discuss the basics of Set Theory. I will assume that you are already familiar with set theory. If you wish to recall things in greater depth then what is covered in this module, I suggest you look back at school text books class 11 class 12. All the concepts that are needed for this course are covered thoroughly in them.

What is a set? A set is a collection of objects.

This raises several questions, because certain terms in this definition are not defined. For instance; what is a collection? What is an object? We will not dwell into the details or what a collection is or what an object is. But rather we will take it for granted that you have an intuitive grasp of what these terms mean.

The term object does not pose that of a difficulty, that much difficulty simply, because in our course all objects will themselves be sets, but the details are irrelevant to us. Now in the first introductory lecture, I emphasize the lot on rigor and being precise and being thorough.

And now, I am saying that you can be a bit lax with set theory. You might think that this is a slightly contradictory way of approaching the subject, but let me make some remarks now. So, as to justify our approach, mathematics is about proving theorems. A theorem is nothing, but a statement that asserts something is true of some object.

How do we prove something is true of some object? Well, we use the properties of that object and deduce that something is true of that object. If you have ever tried to explain something to a very young kid, maybe in 1st standard or 2nd standard; you would notice that many of them are relentlessly curious. If you explain something, they will ask, why? How?

Now, you get a further explanation which breaks up the given explanation into smaller bits, they will still ask, why? How? You notice that no matter how deep you get, you will always have to assume something or the other in order to explain what is going on. So, this means that there is no possible way to explain everything from scratch. You will have to take some things for granted.

I am going to take set theory for granted. There are other more deeper foundational issues that arise in set theory. These are studied in a dedicated course on set theory which is usually a graduate course and quite difficult. So, I will take it for granted that you are familiar with the properties of sets. In a later module, I will briefly discuss an axiomatic approach to set theory where, instead of trying to define a set we list some axioms that sets are supposed to satisfy.

This is an alternate approach which makes perfect sense. So, enough of preliminary remarks, let us continue with the thing.

A set is a collection of objects. We usually denote sets by uppercase letters A, B, C etc. The objects that are supposed to be members are usually denoted by smaller case letters like x, y, z etc and we typically write $x \in A$. This is to be read as belongs to or is in. So, we typically read it as x is in A or then we can also write $x \notin A$. This is to be read as x is not in A or x does not belong to A.

So, sets are a collection of objects. We will also use several other synonyms for collections. Sometimes we will say class, sometimes we will say family. They all mean the same thing for the purposes of this course. Whereas, in a course on set theory depending on which approach you take they might mean different things, but for us collection, class, family are all the same.

Now, we have talked about what a set is and a notation for sets, notation for membership, but how do you write down a set? Well, the most common way to write down a set is by listing the numbers explicitly.

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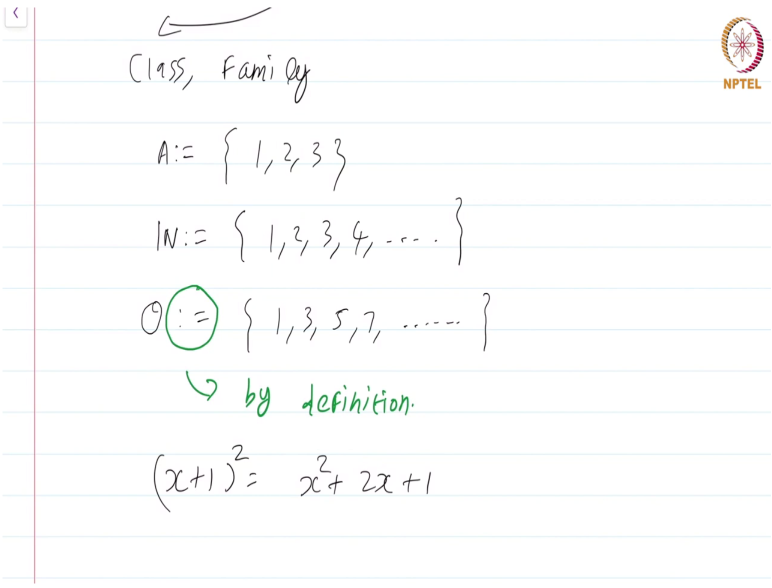

We will write like this. This is the set consisting of three elements 1 2 and 3. Now, we observed that sometimes this notation is completely impractical. For instance, how will you write down the set of natural numbers $\mathbb{N}$? Well, we take several shortcuts. We can write 1 2 3 4 .... Leaving it to the readers understanding, that it is supposed to go on indefinitely and denotes the natural numbers. So, we will rely on intuition whenever possible to reduce the notational quarter.

Sometimes, we can even resort to several deeper shortcuts. For instance, we can define the set $\mathbb{O}$ to be 1 3 5 7 .... Now, this you have to interpret as the set of odd numbers. Now, we will avoid such implicit definitions of sets whenever possible, but it is simply not practical to avoid it at all costs.

They will simply make the notational burden too much. Now, one more thing you would have noticed. When I defined this set $\mathbb{O}$, the set of odd numbers I put a colon equal to.

Well, this is intentional. This is, I mean for consistency let me just put it for the previous definitions also. What this means is 'by definition'. What do I mean by definition?

Well, I want to distinguish between two types of equalities. If you are familiar with programming languages this should not be too hard to understand. When I write something like $(x+1)^2 = x^2 + 2x + 1$. What I actually mean to say is that the left hand side and the right hand side are logically the same. They are not just different quantities that happen to be equal, they are identical quantities written in different ways.

So, whenever I use equality, I mean logical identity. The object on the left hand side and the object on the right hand side are exactly the same. Whereas, I write $\mathbb{O} :=$ say that the left hand side is defined to be the object on the right hand side. So, rather in programming language terms the left hand side if you think of it as a variable name, I am assigning the right hand side to be the value of the variable.

Whereas, when I write $(x+1)^2 = x^2 + 2x + 1$. What I really mean is that the left hand side and the right hand side are exactly the same quantities. So, whenever I define an object, I will use colon equal to. So, it is clear that something is being defined .

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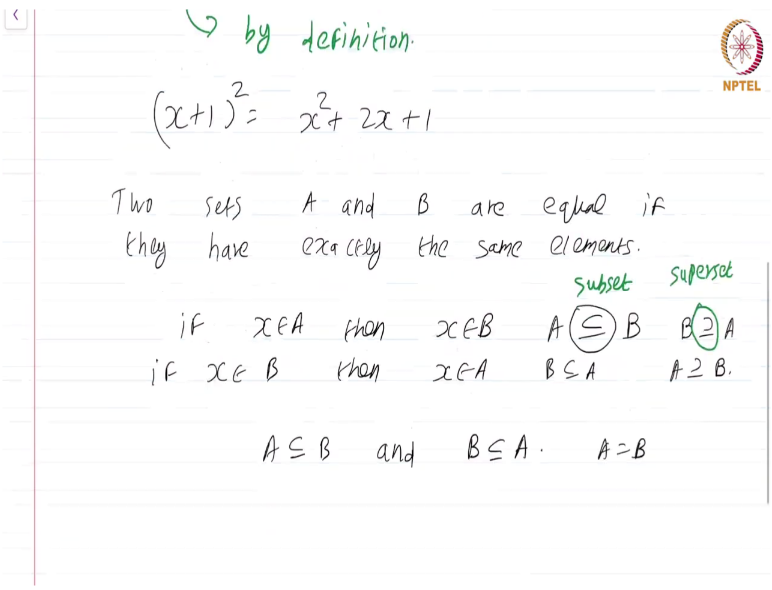

Now, one more notion. Two sets A and B are equal if they have exactly the same elements. Elements are another word for members . So, what this is, saying is if I want to show that two sets A and B are equal, all I have to do is to show that if $x \in A$ then $x \in B$

. Then I have to show that if $x \in B$ then $x \in A$. Now, for the first of these we have a notation in the scenario where if $x \in A$ then $x \in B$, we often write $A \subseteq B$. Sometimes, we also write $B \supseteq A$, that is another notation. The second one is obviously, by analogy is just $B \subseteq A$ or $A \supseteq B$.

To show that A and B are equal. All you need to show is $A \subseteq B$ and $B \subseteq A$. If you show both of these, that means A and B are exactly the same sets , $A = B$

Now, let us introduce some more concepts. We have unions.

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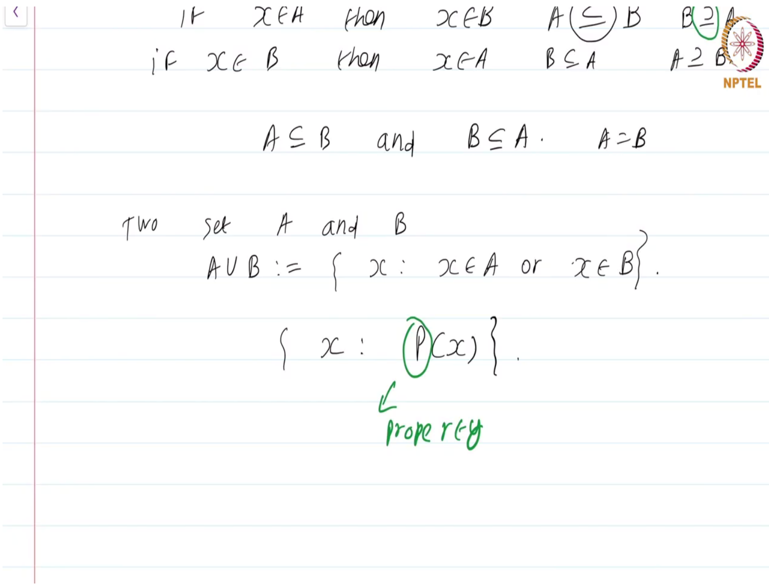

So, if you have two sets A and B. $A \cup B$ is by definition, $\{x : x \in A \quad \text{or} \quad x \in B\}$. Notice that we have used a slightly different notation for denoting this union. Previously to describe a set, we listed explicitly every single member of that set. This is hopelessly inefficient. Instead, most scenarios will require using properties to define sets.

In a future module on logic we will talk about properties, but basically we will write a set as x such that some property is true of x ($\{x : P(x)\}$). Now, the question arises to what set does this property apply to? The property P is always defined on some set. Members of that set could either have that property or not have it.

So, inessence what we are doing is, we are selecting a subset of some set on which the property is defined. We are selecting those members that satisfy the property and putting that as a subset. So, when more precision is required, we will often mention this set explicitly. Instead of just writing x such that property of x is true, we will sometimes denote the set on which the property is defined as S.

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$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

$$\{x \in S : P(x)\}.$$

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

$$A \setminus B := \{x \in A : x \notin B\}.$$

And write the notation as $\{x \in S : P(x)\}$. This is to be read as the subset of those elements of S such that the property P holds for that element x. Now, what set is the property that we are considering in the definition of the union defined over? I have defined $A \cup B$ as a set of all x such that $x \in A$ or $x \in B$.

So, belonging to A or belonging to B is the property, but it is not clear where the property is defined, which set the property is defined. This is not a problem because one of the axioms of set theory that we will see later on specifies that the union of two sets itself is a set. So, in our naive approach to set theory in this module, I am just defining the union this way. In the future module on axiomatics, you will see precisely what is happening.

Similarly, we have A intersection B, $A \cap B$.

This is by definition, $\{x : x \in A \quad \text{and} \quad x \in B\}$. So, intersection is the common elements of two sets and we also have $A \backslash B$, set difference. This is $\{x \in A : x \notin B\}$. So, this is the set difference. Usually to understand these we use conceptual diagrams called Venn diagrams.

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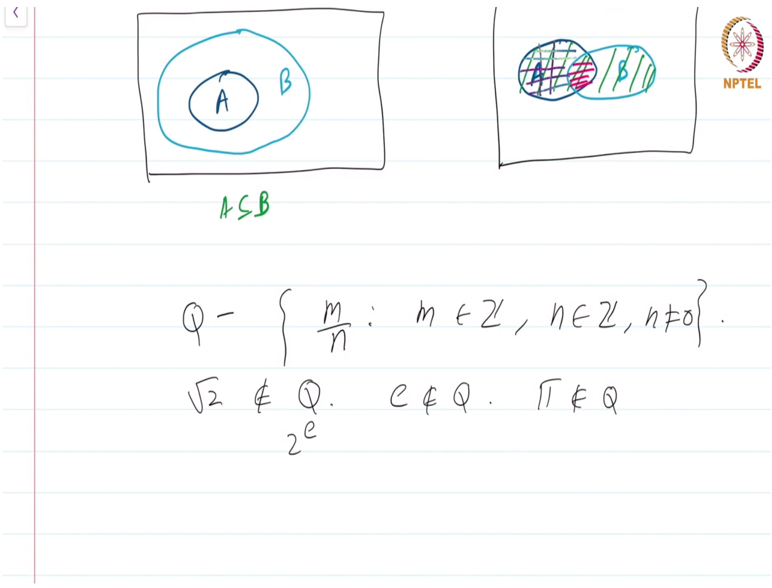

So, let's draw some Venn diagrams for the various concepts we have introduced. If this is the set A, $A \subseteq B$ just means that B is the larger set and A is fully contained in B. This is for $A \subseteq B$. This is the Venn diagram.

Now, for $A \cup B$ and $A \cap B$ and $A \backslash B$, you can illustrate them in the same figure in the following manner. Let us take this to be A and this to be B. So, this is A this is B. This part, this entire thing is the union. The intersection is the common part and $A \backslash B$ or A set difference B supposed to be this; oh sorry, it is just this part; B part is not there. So, this Venn diagram illustrates the concept of union, intersection and complement.

Now, I will not bore you further with further details about set theory, but let me make a remark. We had talked about sets being defined by properties and sets like. We have used words like or and. What do all these mean? Well, I am taking it for granted that you are familiar with the meanings of these terms from your class 11 and class 12 knowledge of set theory.

We will slightly study this in a bit more detail, in the next few modules on logic and proofs where I will clarify what a property means. I will clarify what or, and mean. And, also I will clarify what this if then also means. Logic is a precise language in which mathematics is conducted.

So, it is very good to know at least the rudiments of logic. Again you can take several courses in logic. It is such a vast topic. So, we will have time only to cover the bare minimum. One more remark; when you are defining sets by properties, not all properties are allowed. But let's take the common sets that you already know.

We know the set $\mathbb{Q}$. This is supposed to be $\frac{m}{n}$ such that; $m \in \mathbb{Z}$, $n \in \mathbb{Z}$ and $n \neq 0$. This is the collection of all rational numbers, numbers which are of the form $\frac{m}{n}$. Now, in the previous lecture we showed that $\sqrt{2} \notin \mathbb{Q}$. The proof was slightly involved even though quite elegant and elementary, you needed an idea.

Now, it is also known that the number e which you are familiar with is also not in $\mathbb{Q}$ and the proof is harder. Even harder is the proof that $\pi \notin \mathbb{Q}$. And there are some numbers, for instance 2^e, it is not known whether 2^e is an element of $\mathbb{Q}$ or not an element of $\mathbb{Q}$.

So, when I say a set is a collection of objects. I do not require that the definition of the set gives me an algorithm for deciding whether a particular quantity, a particular object is an element of the set or not. What I merely require is that it should be clear that either a particular thing is in the set or is not in the set.

It is, there should be no ambiguity in the definition. There is no necessity to provide an algorithm when you define a set. So, please keep that in mind. So, several times we will define sets which is not even clear, whether that set, whether there are any elements at all.

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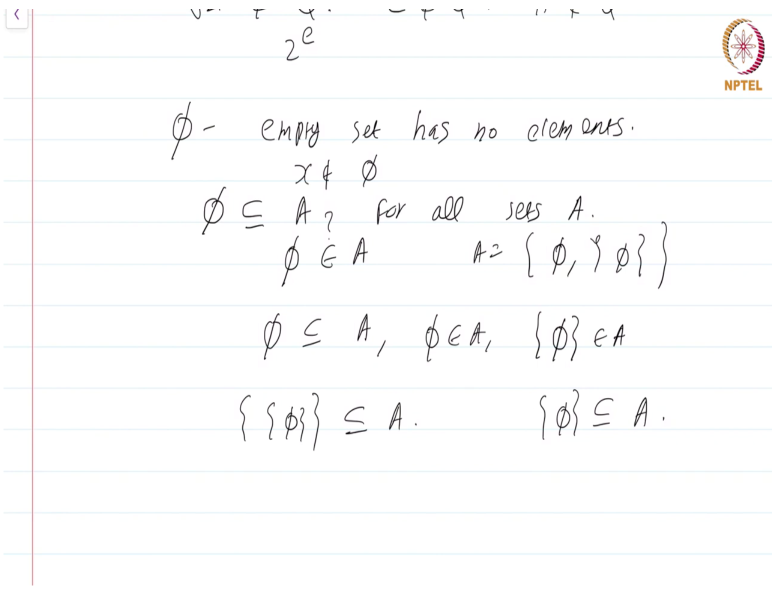

So, to take care of this possibility we introduced the empty set denoted like this (ϕ). This is the empty set. This set has no elements. So, if you take any object x, $x \notin \phi$, the empty set.

Consequently ϕ is a subset. So, again a word of caution; the way I have written ϕ now is wrong; that is the notation for empty set is to be written like this somewhat more digital. So, this ϕ is a subset of A for all sets A; that means, no matter what set you take ϕ is a subset. Why is that so? Well, it will be better if I illustrate this after I talk about some rudiments of logic, but this is what is known as a vacuous statement.

That the fact that the empty set is a subset of A is vacuously true, but be careful it is not true that ϕ is a member of A. This might or might not be true. It depends on the set A. For instance, if the set A looks this $A = \{\phi, \{\phi\}\}$ then $\phi \subseteq A$, $\phi \in A$, $\{\phi\} \in A$. And, $\{\{\phi\}\} \subseteq A$. So, this is very very interesting situation that is happening. Not only that, $\{\phi\} \subseteq A$.

So, this is an extremely comical situation. Please digest this example thoroughly. There is nothing really difficult about the empty set other than getting used to it. This is a course on real analysis and you have just watched a module on basic set theory.