Real Analysis - I

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Lecture – 15.1
The Functions X and X2

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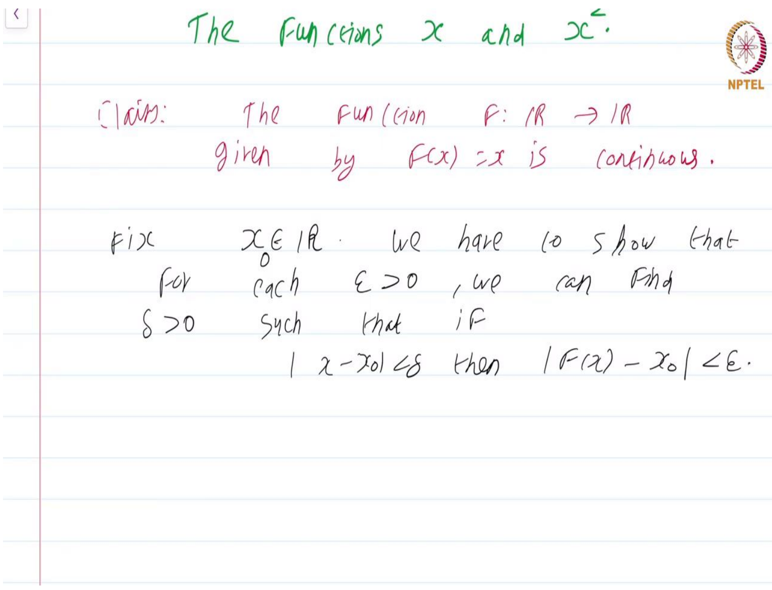

Let us see two simple examples of continuous function.

Claim: The function $F : \mathbb{R} \longrightarrow \mathbb{R}$ given by F(x) = x is continuous.

How do we show this? So, what we are going to do is fix $x \in \mathbb{R}$ or rather to be ultra say fix $x_0 \in \mathbb{R}$.

We have to show that for each $\epsilon > 0$, we can find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|F(x) - x_0| < \epsilon$. I have written $F(x) - x_0$ because $F(x_0)$ is actually just x_0.

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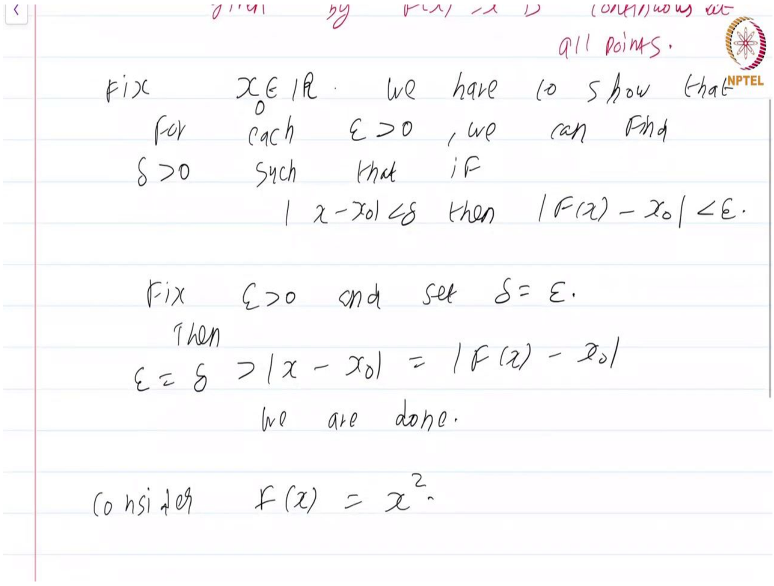

Now, fix $\epsilon > 0$ and set $\delta = \epsilon$, then $|x - x_0| = |F(x) - F(x_0)|$ and this quantity is less than δ which is equal to ϵ . So, $|F(x) - F(x_0)| < \epsilon$ and we are done. That was fairly easy because the choice of ϵ was .I mean the choice of ϵ directly gave the choice of δ without any additional work.

Now, slightly more complicated example; only slightly more. Consider $F(x) = x^2$. I want to claim this is also continuous at all points. So, let me say continuous at all points. Now, how do I show $F(x) = x^2$ is continuous at all points?

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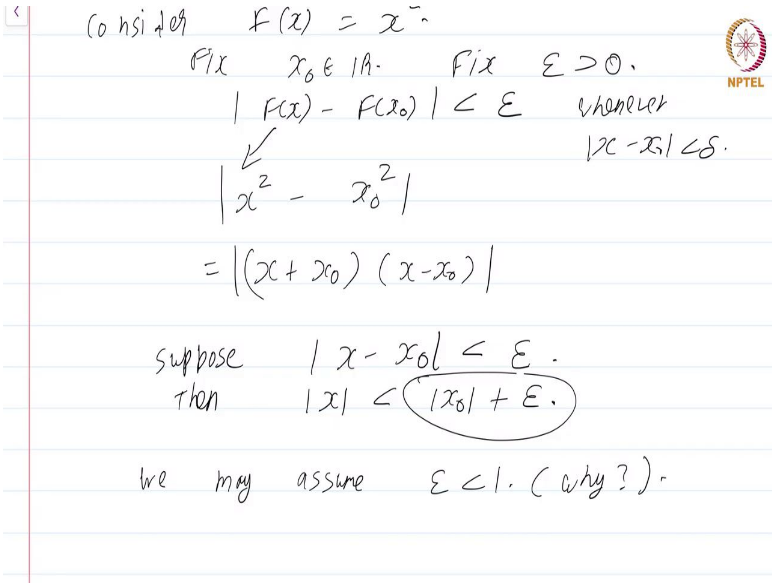

Well, fix $x_0 \in \mathbb{R}$ again and fix $\epsilon > 0$. We must find a corresponding δ such that $|F(x) - F(x_0)| < \epsilon$ and for this appropriate choice whenever $|x - x_0| < \delta$. We have to find that δ. Now, how do we do this?

Well, let us substitute what is happening. $F(x) = x^2$ and $F(x_0) = x_0^2$. So, this is we have to somehow find out a δ such that $|x^2 - x_0^2| < \epsilon$. But $|x^2 - x_0^2|$, this I can write as $|(x - x_0)(x + x_0)|$.

Now, suppose $|x - x_0| < \epsilon$, then $|x| < |x_0| + \epsilon$. If $|x - x_0| < \epsilon$, we get that $|x| < |x_0| + \epsilon$.

Now, I make a statement, we may assume $\epsilon < 1$. Why is this the case? Well, it is the exact same reason why in the definition of the limit of a sequence, we need to consider only $\epsilon < 1$. Think about this and tell me why. You need not satisfy this definition for all ϵ; it is sufficient if you satisfy for all ϵ less than some quantity.

Well, the reason should be fairly obvious to you, even though I have posted as a question. If you can get F(x) to be ϵ close to F(y), you can certainly get it to be 10000 close also because 10000 is greater than 1.

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suppose $|x - x_0| < \varepsilon$.

Then $|x| < |x_0| + \varepsilon$.

we may assume $\varepsilon < 1$. (why?).

$|x| < |x_0| + 1$.

$$
\begin{aligned}
&= |x - x_0| \, |x + x_0| \\
&\le \varepsilon \left(|x| + |x_0| \right) \\
&< \varepsilon \left(2|x_0| + 1 \right).
\end{aligned}
$$

we are done !

So, we may assume $\epsilon < 1$. What does this tell us? It tells us that $|x| < |x_0| + 1$. Now, coming back to this expression, this expression is nothing but $|x - x_0||x + x_0|$, which is less than or equal to $\epsilon(|x| + |x_0|)$ by triangle inequality which is again less than or equal to $\epsilon(2|x_0| + 1)$.

I am just writing the fact that $|x| < |x_0| + 1$. So, in fact, I could have made this less than . Now, we are done. Why are we done? I wanted to find a δ such that if $|x - x_0| < \delta$, then $|x^2 - x_0^2| < \epsilon$.

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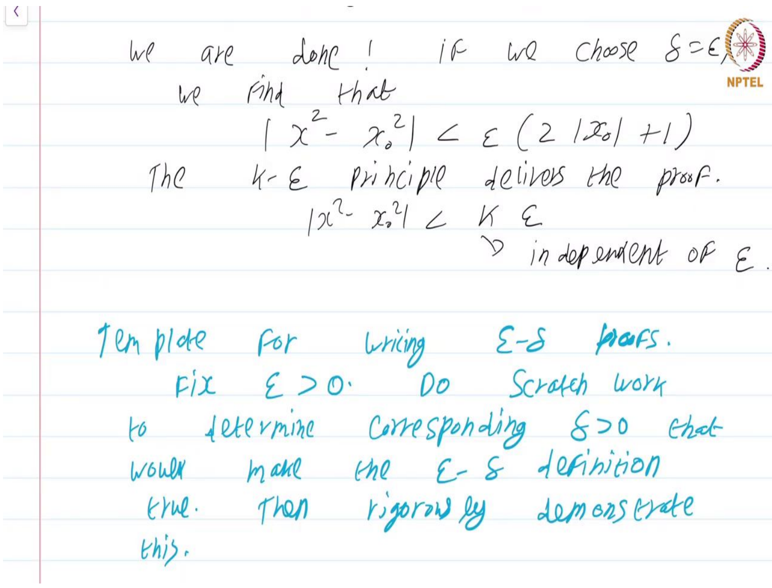

What we have done is if we choose $\delta = \epsilon$, we find that $|x^2 - x_0^2| < \epsilon(2|x_0|+1)$. This is not what we want right. We want it to be less than ϵ . But wait a minute, the $K - \epsilon$ principle.

The $K - \epsilon$ principle delivers the proof. That means, for any fixed ϵ if you can find a bound of the form $|x^2 - x_0^2| < K\epsilon$ and this K is independent of ϵ which indeed it is. That is why we assumed ϵ is less than 1.

If you have this independent of ϵ, then this quantity $x^2 - x_0^2$ can be made arbitrarily small, that was the $K - \epsilon$ principle right. Hence, we are done. The $K - \epsilon$ principle delivers the proof. So, these are prototypes of how to write $\epsilon - \delta$ proofs. The prototype is as follows.

Aim is to show that let me just because it is an important topic, let me give it a subheading. Prototype for rather I would use the word template, template for writing $\epsilon - \delta$ proofs. Fix $\epsilon > 0$. Do scratch work to determine corresponding $\delta > 0$ that would make the $\epsilon - \delta$ definition true, then rigorously demonstrate this.

So, the key is you have to do some scratch work experimentation to find out what the corresponding δ is going to be. So, you must always start such a proof by writing fix $\epsilon > 0$, then give an expression for δ in terms of ϵ .

Sometimes, it will turn out to be independent of ϵ also even if that can happen. Then, rigorously demonstrate that the $\epsilon - \delta$ definition is indeed satisfied.

This is a course on Real Analysis and you have just watched the module on Continuity of the functions X and X squared.