Real Analysis - I
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Lecture - 14.2
Deep Dive into Epsilon-Delta

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A deep into the ϵ-δ definition of continuity.

Let $F : A \to \mathbb{R}$ be a function and $x \in A$.
We say F is continuous at x if
for each $\epsilon > 0$, we can find $\delta > 0$
such that
$|F(x) - F(y)| < \epsilon$
whenever $|x - y| < \delta$ and $y \in A$.

We are about to Dive in Deep for the second time in this course. This time our deep dive will be through the ϵ definition of continuity, which I will state first.

Let $F : A \longrightarrow \mathbb{R}$ be a function and $x \in A$, we say F is continuous at x. If for each $\epsilon > 0$, we can find $\delta > 0$ such that $F(x) - F(y) < \epsilon$, whenever $|x - y| < \delta$ and y is in the set A.

This was one of the three equivalent characterizations of continuity that we saw in the previous module. The first characterization captured our intuitive idea that when x is close to a set B, which is a subset of A, then F(x) must be close to F(B). This is equivalent to that. Therefore, the deep dive is over to see you in the next module.

Hold on a minute; things are not so easy. Notice one crucial difference between the first condition that we saw, which said that whenever x is close to B, F(x) is closed to F(B) and this definition.

Though both are logically equivalent and we have seen a long proof of that, there is something that is bothering us like an itch at a place in your back, which you are not quite able to reach. The problem is here there is an ϵ whose role is there in the co-domain. And that ϵ is taking centre stage. First for each $\epsilon > 0$, we can find a $\delta > 0$ such that something happens.

But, the definition of continuity is supposed to be that when x is close to something, something happens, but x being close to something is happening on the domain side not on the co-domain side. So, there is sort of a role reversal happening in this $\epsilon - \delta$ definition. And it is worthwhile to spend some time to understand it, moreover this is the standard definition of continuity that is used in most textbooks. So, it will be useful to understand this at greater length.

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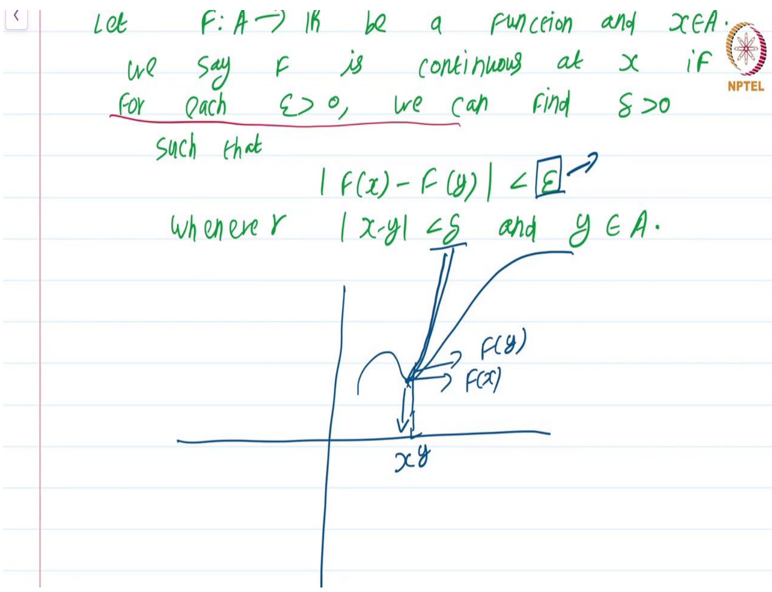

So, let us draw a picture, I have the x - y axis and I draw a function. Now, the key thing is we want to capture what is the meaning of when x is close to a particular point then F of x is also close right. But, the problem is it can happen that the function at this particular point x, increases or decreases rapidly near the point yet without any gaps.

For instance in this at this point it could suddenly just go up, this is still very much a valid function except even when you are near x, F of that point could be somewhat further away, again I have drawn x dangling in the air. So, let me just fix that x is here this is F(x).

But, a point very near if you take y which is very near can still happen that F(y) is somewhat further away and the more highly inflected the graph is the more the you will have to see that, when you are close to x it could still be that the point F of that is somewhat far away.

So, the key is that none of this is breaking continuity; the only issue is we will have to zoom in further to see that the graph is actually continuous. There are only two ways by which a function could fail to be continuous intuitively at least. We will make this precise in a later module. One is that there are huge gaps, or two there are wild oscillations. Neither of which are happening in this particular graph, it is just that the graph is increasing at a very steep high rate.

So, what we want to say is it is not just that whenever y is close to x, F(x) is close to F(y). It is rather saying that you tell me how close you want F(x) to be to F(y) then I will tell you how close y must be to x. So, this ϵ is sort of trying to set a tolerance level.

I want F(y) to be this close ϵ close to F(x) then I will have to return a value of δ, which tells me how close y must be to x in order to satisfy the tolerance level. So, one way to view this $\epsilon - \delta$ definition is as a challenge.

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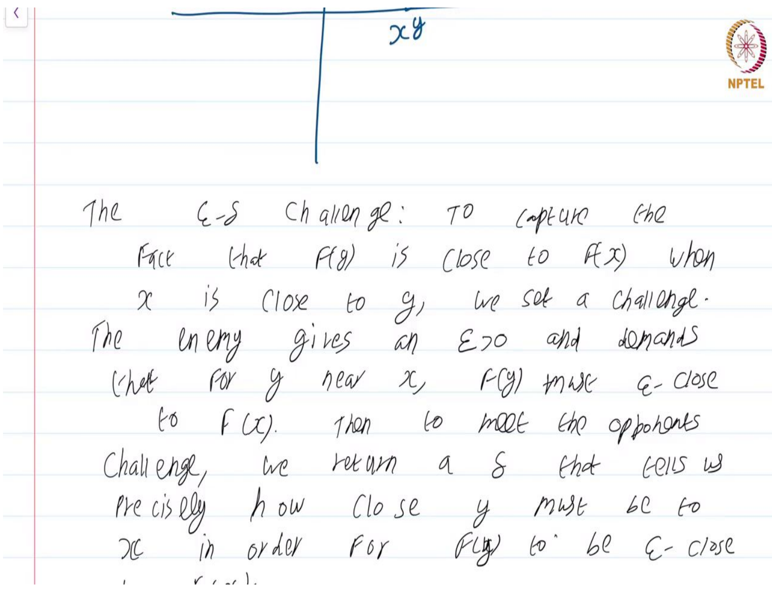

The $\epsilon - \delta$ challenge. So, to capture the fact that F(y) is close to F(x) when x is close to y; we set a challenge. The enemy gives an $\epsilon > 0$ and demands that for y near x, F(x) must be ϵ close to or rather F(y) symmetric, but still F(y) must be ϵ close to F(x) .

Then, to meet the opponents challenge, we return a δ that tells us precisely how close y must be to x. In order for F(x) to be F(y) to be ϵ close to F of x.

So, the abstract definition that we have in mind is that when y is close to x, F(y) is close to F(x). We capture that by saying that, you tell me how close you want F(y) to be F(x) and I will tell you how close y must be to x in order for that to be satisfied.

So, this is going to take care of what happens when the function is either increasing or decreasing very rapidly near the point x, but without any oscillations or without any huge jumps. The function is actually going to be continuous at that point it is going to be clean and nice. However, because it grows so rapidly you cannot expect that just because y is close to x, F(y) will be equally close to F(x).

The relationship between how close F(y) and F(x) are compared to x and y need not be any linear relationship, it might be the case that for F(y) to be ϵ close to F(x), you may have to make y, ϵ^{2000} close to x that can happen because the relationship could be highly non-linear.

This $\epsilon - \delta$ definition captures this in the form of a nice challenge. You should try to formulate this challenge in an analogous way to the epsilon n challenge that you have already seen when you studied limits of sequences. There given an epsilon where you can find a N_ϵ such that if you are beyond N_ϵ. Then, the terms of the sequence will be at max ϵ away from the limit L. So, it is an entirely analogous thing that is happening here for continuity.

So, please think over this, we will solve some examples a couple of modules down the line, where we explicitly find out what delta should be within respect to ϵ. I will give a variety of functions. And you will notice that the relationship between ϵ and δ is not always straightforward. Now, this discussion motivates another definition.

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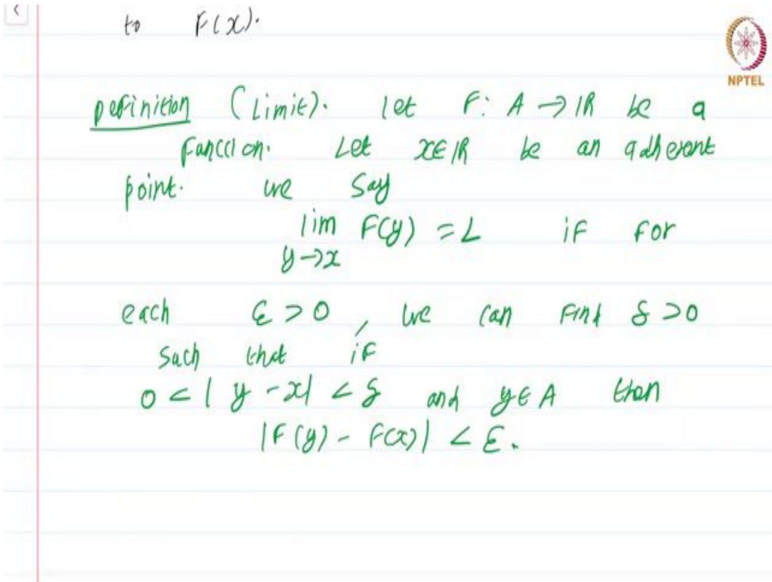

Definition (Limit): Let $F : A \longrightarrow \mathbb{R}$ be a function. Let $x \in \mathbb{R}$ be an adherent point. We say $\lim_{y \to x} F(y) = L$, if for each $\epsilon > 0$, we can find $\delta > 0$ such that if $0 < |y - x| < \delta$, and $y \in A$, then $|F(y) - F(x)| < \epsilon$. So, let me just read out the definition again.

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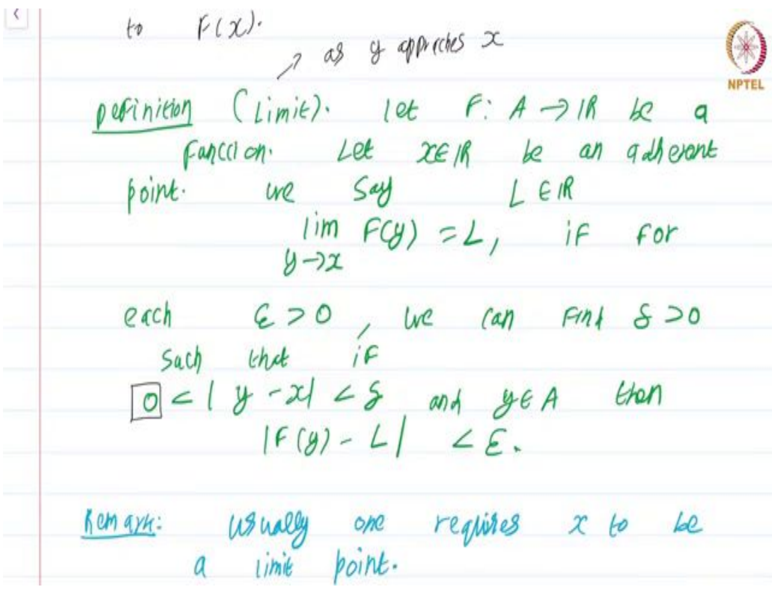

We have a function $F : A \longrightarrow \mathbb{R}$, we pick an adherent point. This point x need not be an element of A , the function F need not be defined at the point x.

We say $\lim_{y \to x} F(y) = L$ and L is a real number. If for each $\epsilon > 0$, we can find a $\delta > 0$ such that if $0 < |y - x| < \delta$ and $y \in A$, then $|F(y) - L| < \epsilon$. Only difference is we are specifying this, $0 < |y - x| < \delta$. The reason is this is the definition of limit as y approaches x .

If I did not put this $0 < |y - x| < \delta$. And if it happens that $x \in A$, then you can just see that there is something going to go wrong. We do not want to consider what happens at x in the definition of limit. We want to only consider what happens, as you approach or tend to this point x. You do not want to consider the value at that particular point for considering limits.

So, this is the notion of limit as y approaches x. Now, let me just make a remark because this is somewhat important.

Remark; usually one requires x to be a limit point not just an adherent point.

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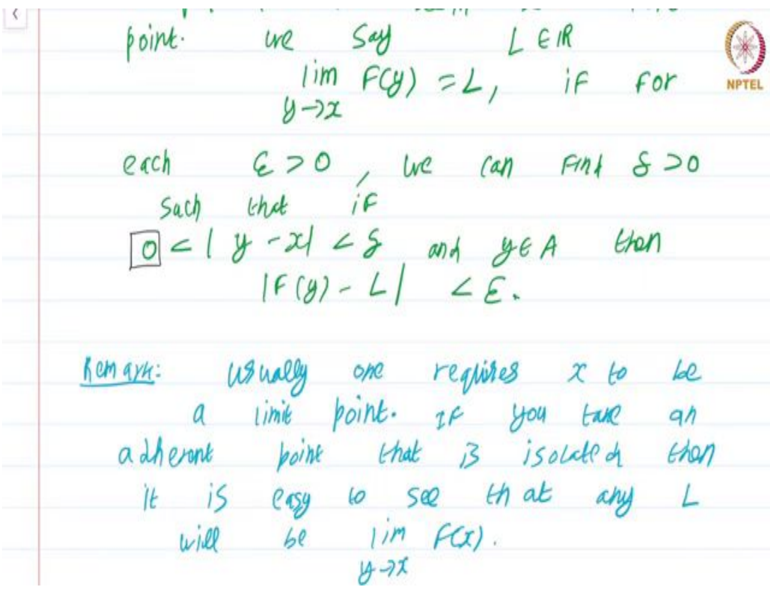

Because, if you take an adherent point that is isolated, then it is easy to see that any L will be $\lim_{y \to x} F(y)$. This extreme weirdness can happen, when you take an isolated point.

Well just think about what is happening. If you take an isolated point of the set A; that means, nearby there are no points of the set A, that is there is some neighbourhood of this point, which is not an element of the set A. That means, this definition when you say, we can find $\delta > 0$ such that if $0 < |y - x| < \delta$ and $y \in A$, there is no such candidate point $y \in A$.

There is no such candidate point $y \in A$, therefore, this definition will be vacuously satisfied for any L. So, we will also do the same, even though I have written adherent points let me just change it to be a limit point.

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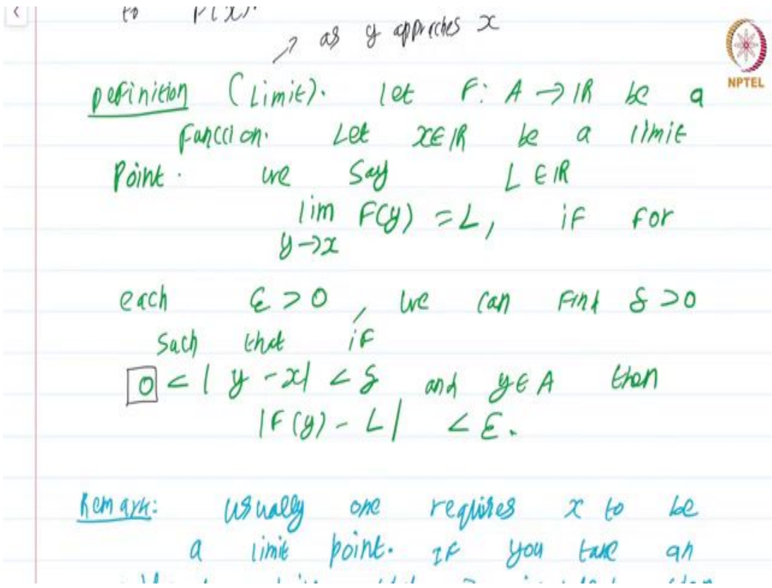

I just wanted to illustrate what will go wrong as a limit point. So, I hope the definition of limit somewhat makes sense to you from our deep dive on the $\epsilon - \delta$ definition of continuity. The usual approach is to first define limits and then define continuity, but continuity is the more fundamental notion.

So, we shall explore both these definitions by giving a huge collection of examples that should reinforce what is happening. As before after you see these examples please revisit the previous module and this module watch it once again, till you have thoroughly digested the definitions of continuity and limits.

This is a course on real analysis and you have just watched the module entitled a deep dive into the epsilon delta definition of continuity.