Real Analysis - I
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Lecture - 14.1
Definition of Continuity

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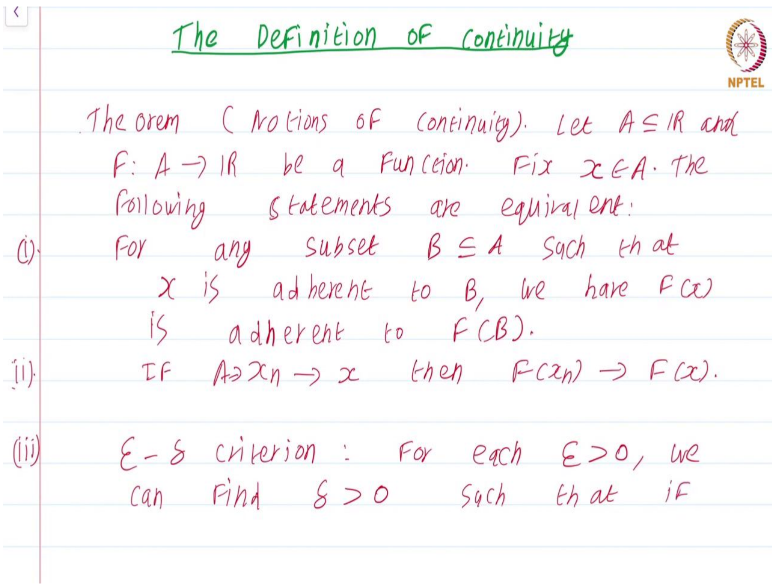

At last we come to the definition of Continuity. This module is probably the most important one in this entire course. So, I suggest you turn off Whatsapp, grab a cup of your favorite beverage and concentrate for the next 30 to 40 minutes. I am going to state a theorem that characterizes various notions of continuity.

Theorem (Notions of continuity) Let $A \subset \mathbb{R}$ and $F : A \longrightarrow \mathbb{R}$ be a function. Fix x in A the following conditions or rather the following statements are equivalent.

(i) For any subset B of A, such that x is adherent to B. We have F(x) is adherent to F(B), if you recall this is almost exactly the informal notion of continuity.

That was third stated earlier with one difference. The word in quotes “close” has been replaced by the word adherent.

(ii) If x_n in A converges to x, then $F(x_n)$ converges to F(x).

(iii) $\epsilon - \delta$ criterion, which is the reason for nightmares for thousands and thousands of math undergraduate students throughout the world.

Let me state this precisely. The $\epsilon - \delta$ criterion is as follows. For each $\epsilon > 0$, we can find $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|F(x) - F(y)| < \epsilon$.

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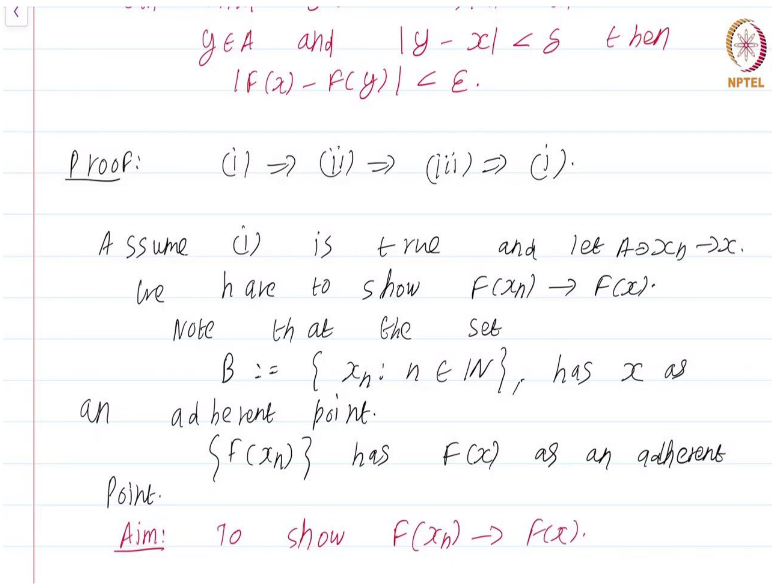

Now I have stated three statements and I am saying that the following are equivalent. That means, whenever one of the statements is satisfied, the other two statements are also automatically satisfied. Now, I call these notions of continuity because these are three good ways of trying to formulate what a function F being continuous at the point x really means.

The first one we have already explored at some length. In a moment, it will be clear to you that the second one is just a slight variant of the first, but the third one deserves some comments as that is the standard definition of continuity given in most places. So, I will dedicate an entire module to the third one.

Now what I am going to do is, I am going to prove that all three statements are equivalent, then I recommend that after watching the next few modules you revisit this module yet again. In fact, I would strongly recommend that this module be watched at least three times to digest what exactly is going on.

Proof: The standard way of showing that three statements are equivalent is to show that $(i) \implies (ii) \implies (iii) \implies (i)$, that is, form a complete circle. Let us see how we can do that. Let us look at the first statement, for any subset B of A, such that x is adherent to B. We have F(x) is adherent to F(B). The second statement we have to show that if $x_n \in A$ is a sequence that converges to x, then $F(x_n)$ must converge to F(x).

So, assume (i) is true and let $x_n \in A$ converge to x. We have to show $F(x_n) \to F(x)$. So for all I have done is rewrite the thing that I am supposed to do. Note that this x_n is the set $\{x_n\}$. So, let me say be ultra precise what this set is. Note that the set $B := \{x_n : n \in \mathbb{N}\}$.

I am considering all the points in the sequence (x_n). Note that the set B as x as an adherent point. Well, that is because $x_n \to x$, that is the very definition of an adherent point. That means the set $F(x_n)$ has F(x) has an adherent point, that is what our assumption (i) is.

Whenever you have a set that has x as an adherent point F, the F(B) should have F(x) as an adherent point. In this case, F(B) is nothing, but $F(x_n)$. Our aim is to show that $F(x_n) \to F(x)$. So, what we are going to do is the following. So, let me just write down the aim, so that it is in our mind to aim to show $F(x_n) \to F(x)$.

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We will show that any subsequence $F(x_{n_k})$ has a further subsequence that converges to $F(x)$.
Start with a subsequence $F(x_{n_k})$
Either
(i) $F(x_{n_k}) = F(x)$ infinitely often. Then we are done.

(ii) $F(x_{n_k}) = F(x)$ for only finitely many k. Observe that in this scenario, $F(x)$ is infact a limit point (why?). That means there is a sequence $y_m \in \{F(x_{n_k})\}$

So, now how we are going to do this is a slightly roundabout way. We will show that any subsequence $F(x_{n_k})$ has a further subsequence that converges to F(x).

Recall that in the chapter on sequences in series we had shown that if a sequence has the property that any subsequence has a further subsequence that converges to a particular point, then the whole sequence converges to that particular point. So, we are going to exploit this result.

So, start with the subsequence $F(x_{n_k})$. Now there are two possibilities either number 1) $F(x_{n_k}) = F(x)$ infinitely often. So, that means for infinitely many k, $F(x_{n_k}) = F(x)$ then we are done. I am not even going to bother writing down how we are done. I am going to leave it to you.

So, the second case is $F(x_{n_k}) = F(x)$ for only finitely many k, it is a pure dichotomy. This is the case we have to deal with. So, if $F(x_{n_k}) = F(x)$ for only finitely many k, then you cannot extract a subsequence in the easy manner. That happened in case 1, when there were infinitely many k's for which $F(x_{n_k}) = F(x)$, you will have to do something else.

Observe that in this scenario, F(x) is in fact the limit point. Why? I am not going to tell you why, it is up to you to figure this out. That means there is a sequence y_m.

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many k. observe that in this scenario, F(x) is infact a limit point (why?). That means there is a sequence $y_m \in \{F(x_{n_k})\}$ such that the y_m's are distinct and $y_m \to F(x)$.

Observe that each y_m is of the form $F(x_{n_k})$ for some k.

Choose $z_1 = F(x_{n_{l_1}}) = y_1$.

Now as the terms in the sequence y_m are all distinct there must be some y_s, $s > 1$ such that $y_s = F(x_{n_{l_2}})$ and $l_2 > l_1$.

Let us say I use a different subscript y_m coming from the set of elements $F(x_{n_k})$, such that the y_m's are distinct and $y_m \to F(x)$. This is the very definition of a limit point. Now I am going to use this y_m to construct the required subsequence. So, observe that for each y_m is of the form $F(x_{n_k})$ for some k. I do not know what that is for some k excellent.

Now this is how I am going to construct the subsequence. Choose what I will do is the following. I will construct a subsequence of (y_m) which itself would be a subsequence of as well.

I will produce a sequence let us say some z_l, which will be a subsequence of both y_m as well as . How do I do that? Well choose $z_1 := F(x_{n_l}) = y_l$.

Now, as the terms in the sequence (y_m) are all distinct. There must be some there must be some $y_s,\ s > 1$, such that $y_s = F(x_{n_{l_2}})$ and l_2 is strictly greater than l_1. Note the terms of the sequence y_m are all distinct. So, it cannot happen that the terms of the sequence keep oscillating all the way less than $F(x_{n_{l_1}})$, where I mean to say it cannot happen.

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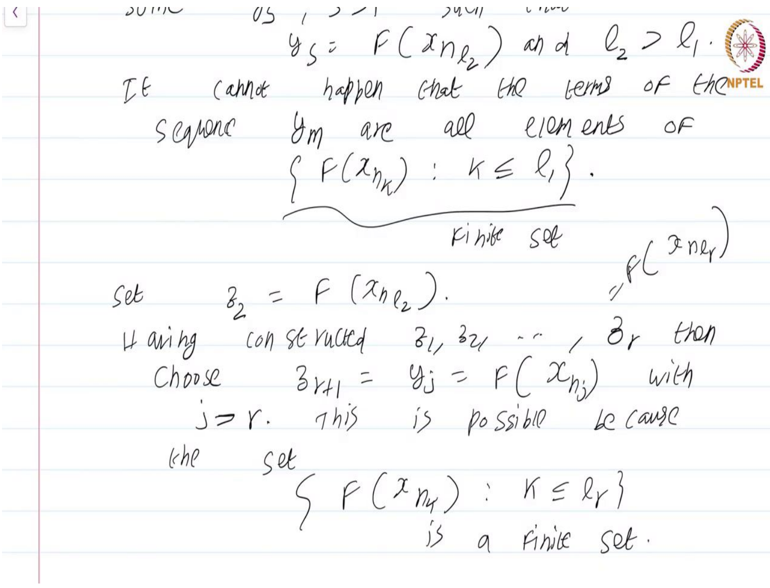

It cannot happen that the terms of the sequence y_m are all elements of the set $F(x_{n_k}),\ k \leq l_1$, then there will have to be repetitions because this is a finite set. At some point y_s has to be $F(x_{n_{l_2}})$ and $l_2 > l_1$ set. This z_2 to be nothing, but this $F(x_{n_{l_2}})$.

So, having constructed z_1, z_2 up till let us say z_r, then choose and this z_r. I am going to call it $F(x_{n_r})$, then choose z_{r+1} to be some y_j equal to $F(x_{n_j})$ with j > r.

This is possible because the set $F(x_{n_k})$, k less than or equal to, let me just change the notation slightly. Here, I need to use two subscripts, because it is a subsequence. So, this is n_l . $r_k < l_r$ is a finite set. So, this is a bit complicated.

What is happening, mainly because there is a notational over-burden. That is why I want you to watch this module at least once or twice more after you come to grips with the basic facts about continuity in the next few modules, but basically what we have done is to somehow construct this subsequence of both y_s as well as $F(x_{n_k})$.

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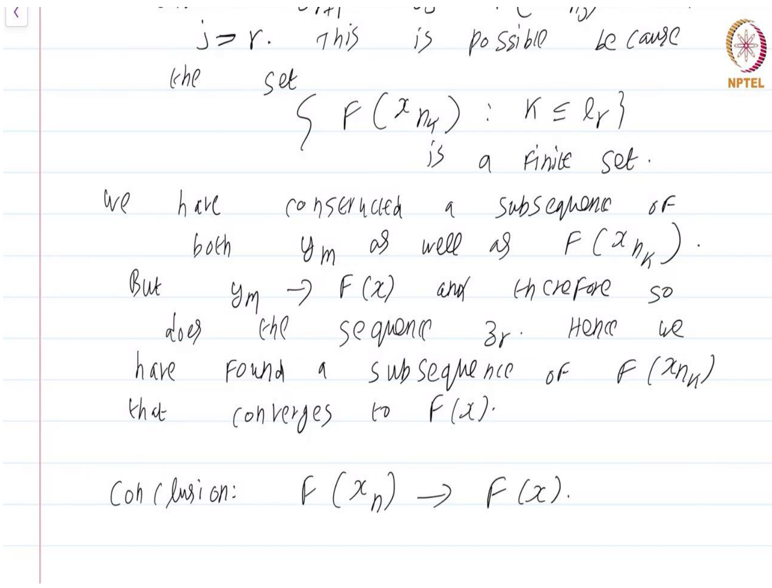

Now, we have constructed a subsequence of both y_m as well as $F(x_{n_k})$. We have sort of a common subsequence of both these sequences , but $y_m \to F(x)$ and therefore, so does what I call this. Well let me just call it the sequence (z_r).

So, does the sequence (z_r) hence, we have found a subsequence or rather a subsequence of $F(x_{n_k})$, that converges to F(x). So, in both scenarios we have found a subsequence of $F(x_{n_k})$ that converges to F(x). So, the conclusion is $F(x_n)$ must converge to F(x).

So, this was a somewhat long and arduous proof, but the basic idea is really easy. We deal with two cases. The first case in which you do not have to do much work. You can find a very simple subsequence that converges to F(x), then in case you cannot find a simple subsequence that converges.

You are forced to conclude that F(x) is a limit point that gives us a sequence of unique terms. Using this unique term sequence (y_m), you construct a common subsequence and that does the trick. So, please go through this proof carefully once more. So, what have we achieved? We have now shown that (i) implies (ii).

That means, if it is true that whenever x is an adherent point of a subset B we have F(x) is an adherent point of F(B), then for all sequences x_n converging to x, $F(x_n)$ must converge to F(x) excellent.

Now $(ii) \implies (iii)$, that is, whenever you have a sequence x_n converging to $x \in A$, $F(x_n)$ must converge to F(x).

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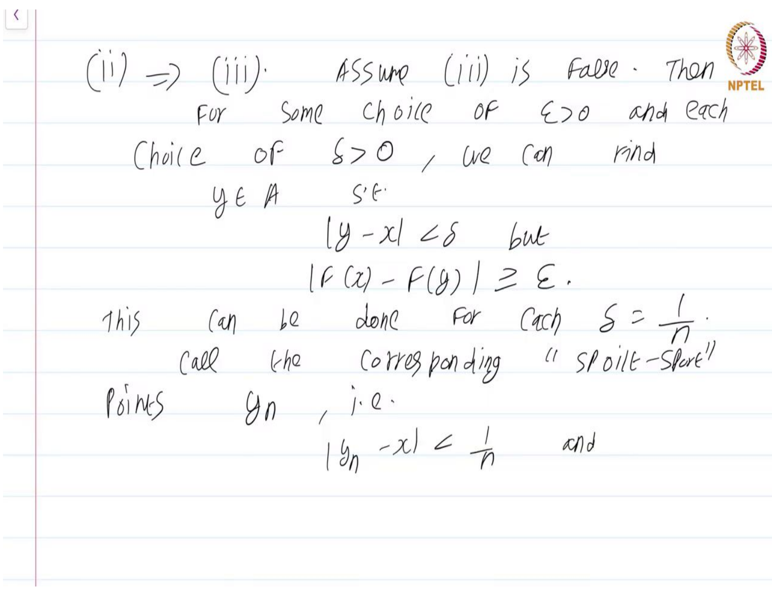

So, now we are in the case (ii) implies (iii). Now what does (iii) say, let me read out (iii) carefully. (iii) says for each $\epsilon > 0$, we can find $\delta > 0$ such that, if $y \in A$ and $|y - x| < \delta$, then $|F(x) - F(y)| < \epsilon$.

Now, this is very reminiscent of the definition of the convergence of a sequence. So, what do we do? Well, think about this for a second how can (iii) fail to be true. The only way that (iii) can fail to be true is for some $\epsilon > 0$ we cannot find $\delta > 0$, such that something happens.

That means, for some $\epsilon > 0$ and for all $\delta > 0$, there must be a point y in A and $|y - x| < \delta$, such that $|F(x) - F(y)| \geq \epsilon$.

So, let us see how to fix that. Assume (iii) is false, then for some choice of $\epsilon > 0$, and all choices of $\delta > 0$, we can find rather than writing all choices. So, for some choice of $\epsilon > 0$ and each choice rather than writing all I will use each. It will clarify things better.

Each choice of $\delta > 0$ we can find y in A such that $|y - x| < \delta$, but $|F(x) - F(y)| \geq \epsilon$, no delta works. So, for each δ there must be some y that place spoilt sport. That place spoilt sport. Now how do you do this? Well this can be done for each $\delta = \frac{1}{n}$.

So, what this means is, we get the corresponding points call the corresponding ''spoilt-sport'' in quotes ''spoilt –sport'' points y_n, that is $|yn - x| < \frac{1}{n}$.

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Choice of $\delta > 0$, we can find
$y \in A$ s.t.
$|y - x| < \delta$ but
$|F(x) - F(y)| \geq \varepsilon$.
This can be done for each $\delta = \frac{1}{n}$.
call the corresponding "spoilt-sport"
points y_n, i.e.
$|y_n - x| < \frac{1}{n}$ and
$|F(x_n) - F(x)| \geq \varepsilon$.
Clearly $y_n \to x$. But $|F(x_n) - F(x)| \geq \varepsilon$ $\forall n$.
This means $F(x)$ is NOT an adherent point of $\{F(x_n)\}$.

And $|F(x_n) - F(x)| \geq \epsilon$. Now, clearly $y_n \to x$. That is how y_n's were constructed $|y_n - x| < \frac{1}{n}$. Therefore, $y_n \to x$, but $|F(x_n) - F(x)| \geq \epsilon$ for all n, this means F(x) is not an adherent point of the set $F(x_n)$. Just a moment I got ahead of myself. I tried to prove one from this. Well, I do not want to prove (i).

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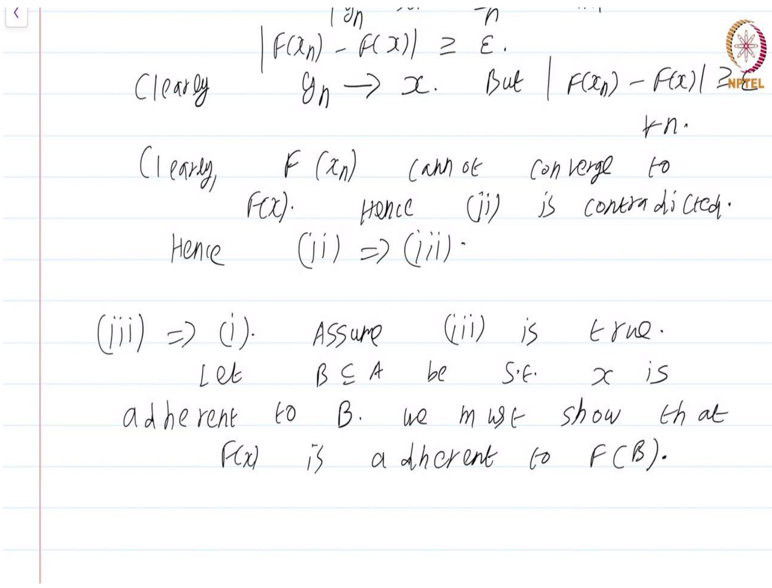

I want to prove ok, but clearly $F(x_n)$ cannot converge to F(x). I tried to contradict (i). In fact, (i) is contradicted, but I want to prove that (ii) is contradicted. That is also good enough. Clearly, $F(x_n)$ cannot converge to F(x), hence (ii) is contradicted.

Hence $(ii) \implies (iii)$. Finally, we have to prove that (iii) implies . That means, we are assuming the $\epsilon - \delta$ criterion we have to show that whenever we have a set subset B such that x is adherent to B, we must have F(x) is adherent to F(B).

Let us prove that assuming (iii) is true. Let $B \subset A$ be such that x is adherent to B. Now we must show that F(x) is adherent to F(B). Again, I am just writing down what I must achieve.

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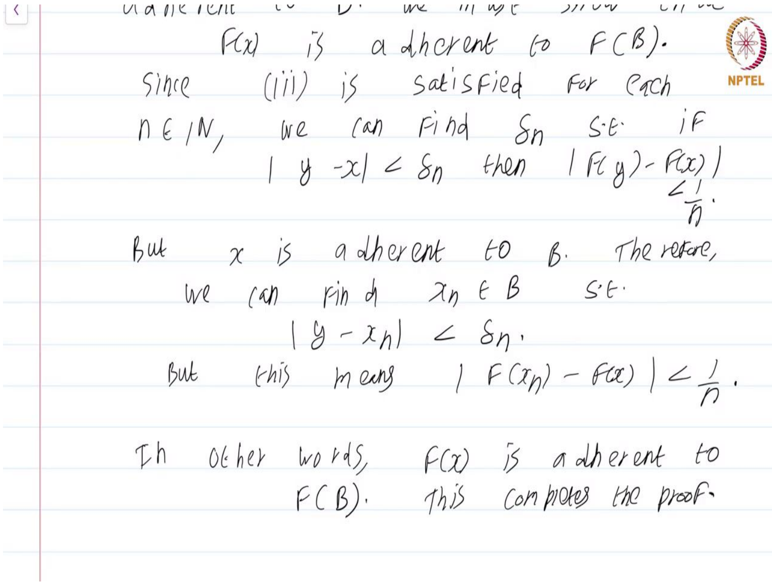

Now, since (iii) is satisfied, for each n in the natural numbers, we can find δ_n such that if $|x_n - x| < \delta_n$, then $|F(x_n) - F(x)| < \dfrac{1}{n}$, (iii) says that for each $\epsilon > 0$, there is a δ such that something happens.

What I am doing is, I am taking $\epsilon := \dfrac{1}{n}$ and taking the corresponding $\delta = \delta_n$. So, I made a mistake such that if $|y - x| < \delta$, then $F(y) - F(x)| < \dfrac{1}{n}$. I have not really constructed an x_n yet, but x is adherent to the set B.

Therefore, we can find $x_n \in B$, such that $|y - x_n| < \delta_n$, right simply because this is one of the properties of adherent points, but this means $|F(x_n) - F(x)| < \dfrac{1}{n}$. That is what the previous condition that is what δ_n, that is how δ_n was chosen. In other words, F(x) is adherent to F(B), fine.

So, this completes the proof. I hope you have enjoyed the cup of coffee and some difficult mathematics. So, this is a long proof as always please go through the notes where things are done. This is a lecture, so I cannot just copy the notes and put it in the lecture. I am lecturing from my mind.

So, best to go through this as well as go through the lecture; the lecture notes where things are done more precisely without any slip ups on my end. I am sure with the help of this lecture as well as the notes you will be able to understand why $(i) \implies (ii) \implies (iii) \implies (i)$ and all three definitions here are equivalent.

So, let us end the lecture with our reward. Our reward is at last the central thing, the definition of continuity.

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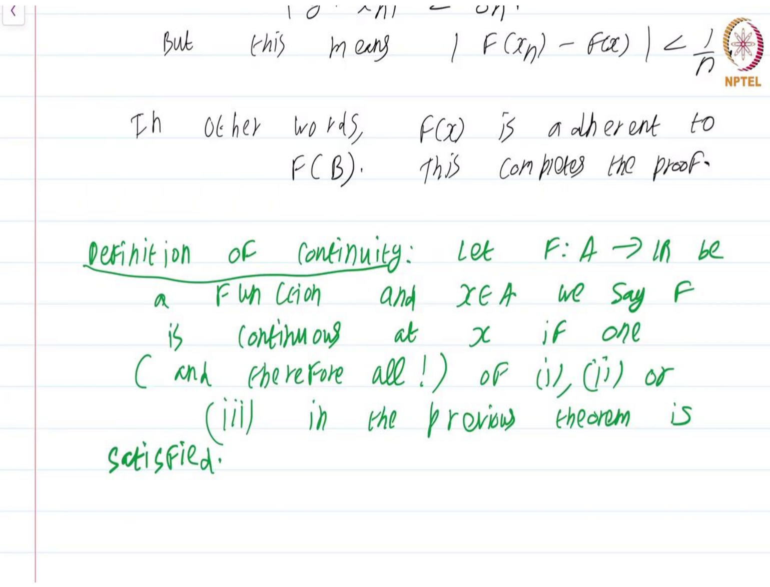

Definition of continuity: Let $F : A \longrightarrow \mathbb{R}$ be a function and $x \in A$, we say F is continuous at x if one and in parenthesis and therefore, all of (i), (ii) or (iii) in the previous theorem is satisfied.

So, please again go through this lecture at least once or twice, do it once again after a few more modules where you see many continuous functions and some theorems about continuity.

So, this concludes this module. This is a course on Real Analysis and you have just watched the lecture on Notions of Continuity.