Real Analysis - I

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Lecture – 13.4
Basic Properties of Open and Closed Sets

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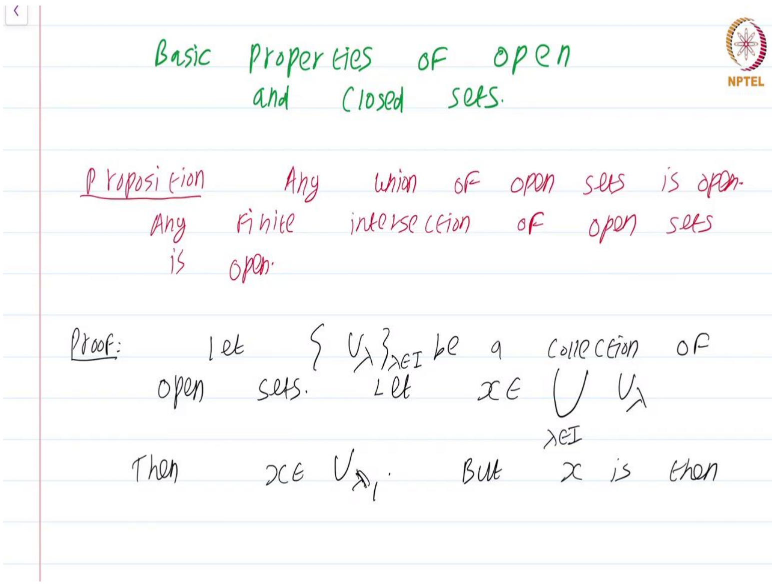

In this module, we will see some Basic Properties of Open and Closed Sets. I immediately begin with a proposition.

Proposition: Any union of open sets is open. Any finite intersection of open sets is open.

Now, the first part I have already hinted how to prove; it is quite easy. But let us for the sake of completeness give a full proof.

Proof: let $\{U_\lambda\}$ be a collection of open sets. Let $x \in \cup_{\lambda\in I} U_\lambda$.

So, I am considering a collection $\{U_\lambda\}$ of sets; its indexed by $0, 25$. It could be countable, uncountable; I don't really care. So, let me just precisely write $lambda$ coming from some indexing set I. So, let $x \in \cup_{\lambda\in I}$, then $x \in U_{\lambda_1}$. Because it is there in the union.

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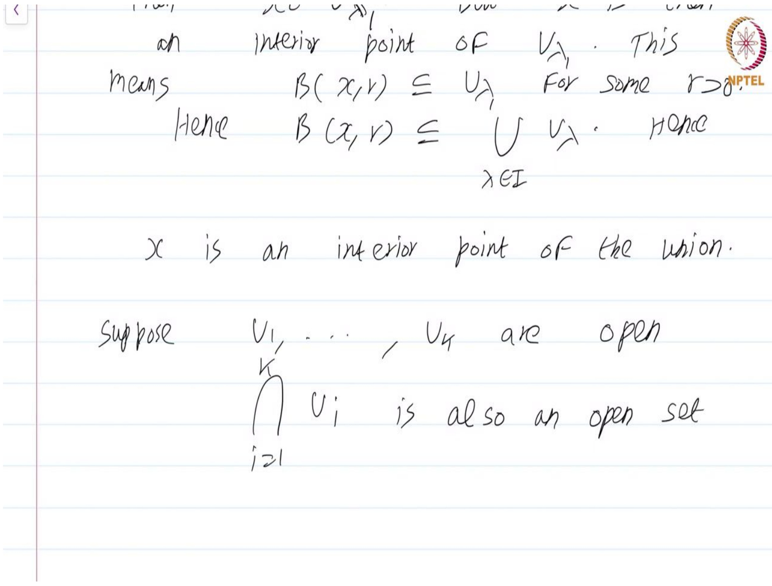

But x is then an interior point of U_{λ_1}. This means $B(x,r) \subset U_{\lambda_1}$ for some r > 0. Hence, $B(x,r) \subset \cup_{\lambda \in I} U_\lambda$. Hence x is an interior point of the union.

Now, suppose $U_1, ..., U_k$ are open, we are on to the second part now. We want to show that $\cap_{i=1}^{k}$ is also an open set.

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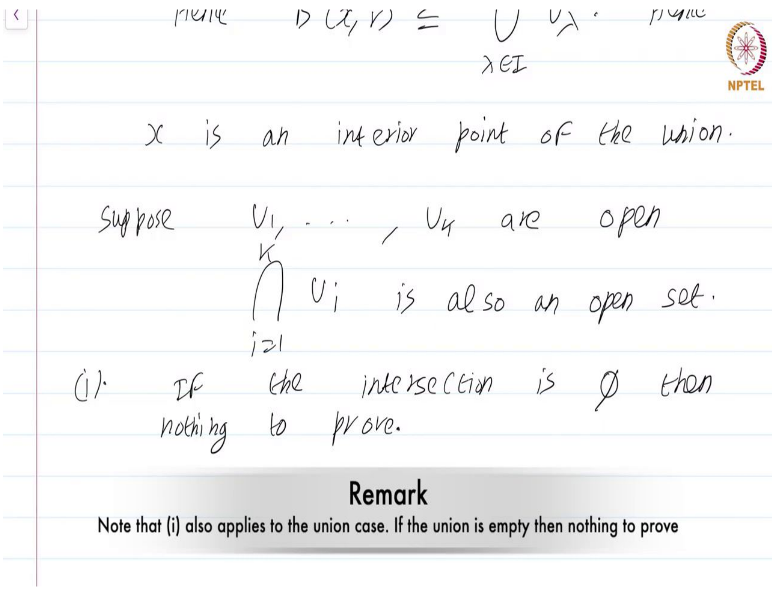

Now, here is the catch. First part if the intersection is empty, then nothing to prove. So, somewhat counter intuitively, the empty set is an open set. Why is that the case? The reason is because the definition of an open set is that every point in that set is an interior point; but the empty set has no points. Therefore, this is vacuously true. So, an empty set is an open set. So, if the intersection is empty, nothing to prove.

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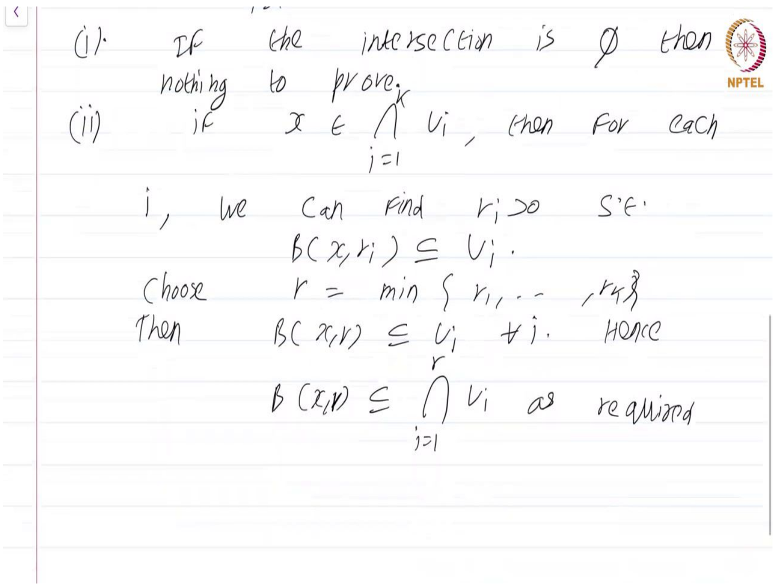

ii) If $x \in \cap_{i=1}^{k} U_i$, then for each i, we can find $r_i > 0$ such that $B(x, r_i) \subset U_i$. The only way by which x could be in each of the U_i's is if x is an interior point of each one of the U_i's simply because each U_i is open.

Because each U_i is open. Corresponding to this U_i, I can find an r_i such that $B(x, r_i)$ is contained in U_i. This is the definition of an interior point and open set. Choose as you can guess $r := min\{r_1, ..., r_k\}$. Then B(x,r) is contained in U_i for all i because it is the smallest of the r_i's. Hence $B(x, r) \subset \cap_{i=1}^{r} U_i$ as required.

We have shown that there is some r such that B(x,r) is contained in the intersection. Therefore, x is an interior point and therefore, the intersection is an open set. So, the next question is what about similar things for closed sets? Is a union of a closed set closed? Is the finite intersection of a closed set of closed sets closed? But before that let us tie up the relationship between closed sets and open sets neatly in the next proposition; the next proposition.

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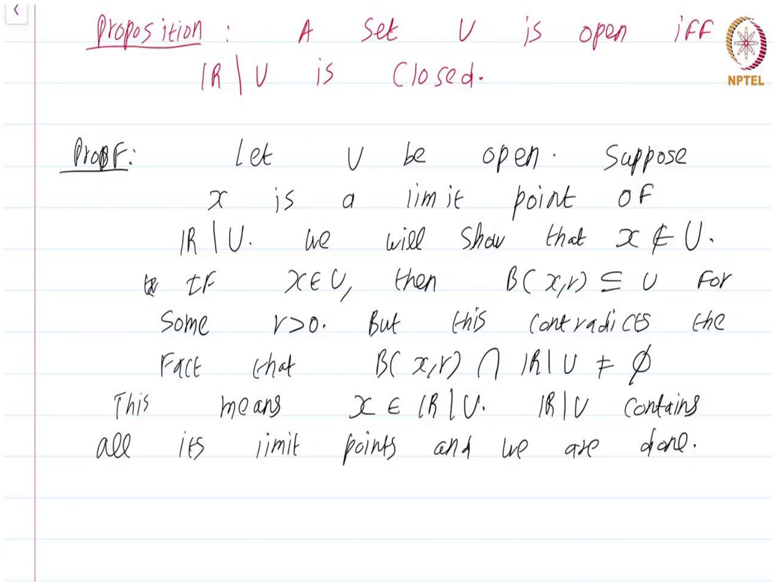

Proposition: A set U is open if and only if $\mathbb{R} \setminus U$ is closed.

So, I remarked earlier that a closed set is sort of dual of an open set and vice versa. That is captured precisely by this proposition which says that set U is open if and only if its complement is closed . How do you prove this proposition?

Proof: let U be open. Suppose, x is a limit point of $\mathbb{R} \setminus U$.. We will show that x cannot be an element of U. Why is that the case?

Well, if $x \in U$, then B(x,r) is a subset of U for some r > 0. Why because U is an open set; every element of U must be an interior point right. But this contradicts the fact that $B(x,r) \cup \mathbb{R} \setminus U \neq \phi$. Because x is an adherent point; it is in fact, a limit point of B(x,r) intersect $\mathbb{R} \setminus U$ must be non-empty for all choices of r. But we have found an r for which we B(x,r) is contained in U.

So, this means x is an element of $\mathbb{R} \setminus U$; no choice. So, what we have shown is $\mathbb{R} \setminus U$ contains all its limit points and we are done. Hey, hold on; did not we say that a set is closed, if it contains all its adherent points; not just all its limit points.

Well, not a problem, we just showed that an adherent point is either an isolated point or a limit point. An isolated point has to be in the set itself. So, in fact, to show that a set is closed, you need not show that all adherent points are there in the set; you can just show all limit points, that is enough.

Because the non-limit adherent points will automatically be in the set, just by the way things have been set up. So, this was fairly straight forward. Now, we need to prove the other side.

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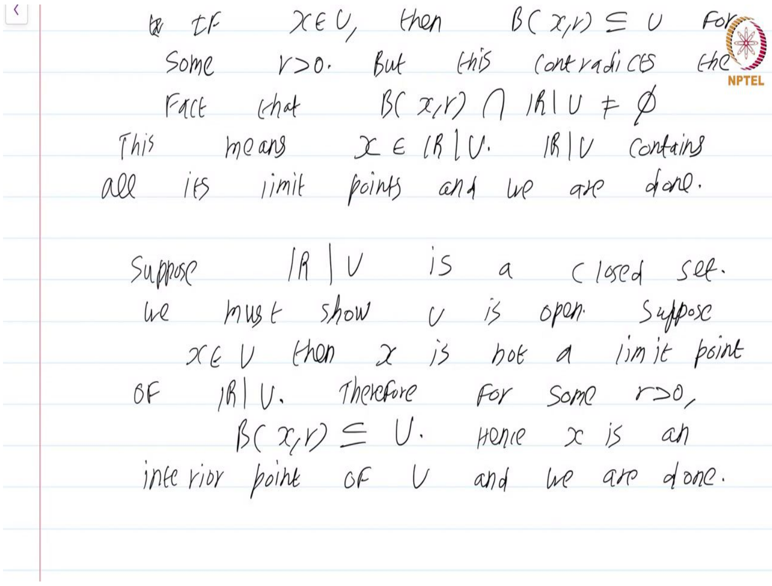

Suppose $\mathbb{R} \setminus U$ is a closed set, then we must show U is open. Suppose, $x \in U$; then x is not a limit point of $\mathbb{R} \setminus U$ because $\mathbb{R} \setminus U$ is closed and therefore, it must contain all its limit points.

Therefore for some r > 0, B(x,r) must be fully contained in U. This is the only way by which x can fail to be a limit point of $\mathbb{R} \setminus U$. Hence, x is an interior point of U; interior point of U and we are done and we are done. So, what we have done is we have shown that a set is open if and only if, its complement is closed. So, they are both dual notions.

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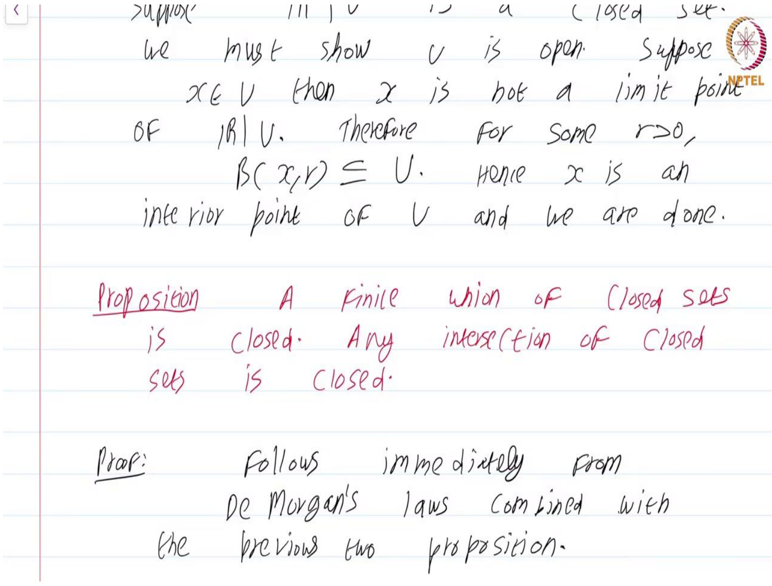

Now, I can immediately write down this proposition;

Proposition:A finite union of closed sets is closed. Any intersection of closed sets is closed.

Proof: Follows immediately from De Morgan's laws, combined with the previous two propositions. I am not even going to elaborate the proof. I am going to leave it to you to figure this out.

So, from the facts about open set, we have immediately got the facts about closed sets . Now, I am going to introduce just two more concepts which are sort of producing closed sets and open set from a given set.

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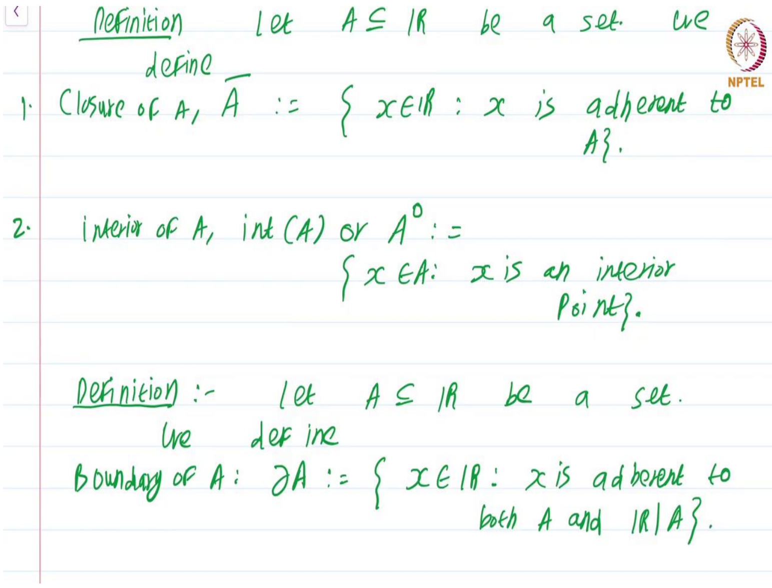

Definition; let $A \subset \mathbb{R}$ be a set. We define $\overline{A}$.This is called closure of A.

1. $\overline{A} = \{x \in \mathbb{R} : x \ is \ adherent \ to \ A\}$
2. Interior of A; denoted int(A) or A°.

I do not like the second notation because some textbooks use that for compliment. I prefer writing Int(A).

$$Int(A) := \{x \in A : x \ is \ an \ interior \ point\}.$$

So, these are two things. I want to define one more thing. I want to give it as a separate definition.

Definition: let $A \subset \mathbb{R}$, we define the boundary of A. This is called boundary of A. So, we want to say that something is a boundary point. Well, this is defined to be

Boundary of A: $\partial A = \{x \in \mathbb{R} : x \ is \ adherent \ to \ both \ A \ and \ \mathbb{R} \setminus A\}$. Something is in the boundary of a set, if it sticks to both the set and the outside of that set. So, this is a very very natural definition. Now, in a proper course on topology, you explore several relationships between all the definitions that we have given.

You will explore some of these in the exercise; but for the time being, this is all I require from basic topology to continue with real analysis. Any other topological concepts that I need will be introduced as and when needed. So, we can now proceed to the definition of continuity, after which we shall study the relationship between topological properties and continuity, how they interact; at that time, I will introduce some more topological notions.

This is a course on Real Analysis and you have just watched the module on basic properties of open and closed sets.