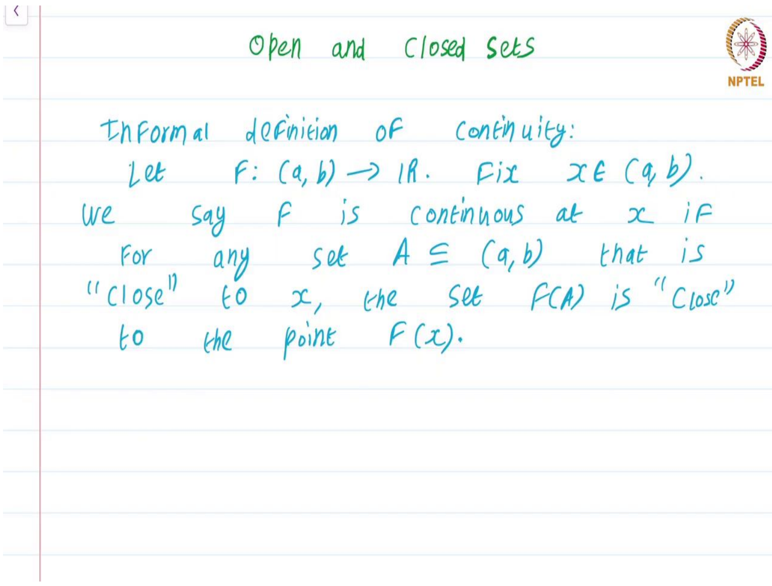
Real Analysis - I

Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 13.2
Open and Closed Sets

(Refer Slide Time: 00:14)



It is customary to begin topology by defining Open Sets. We will be a bit different and define Closed Sets first. The reason for this is that it's easier to motivate why we need to study closed sets in the context of defining continuous functions. From the previous module, where we talked about the role of topology; the following definition should seem very reasonable.

Let me remark that the definition I am about to state is informal and intuitive and you will see that this can be made precise once we have a precise definition of continuity using the tools of topology. So, let me begin with an informal definition of continuity.

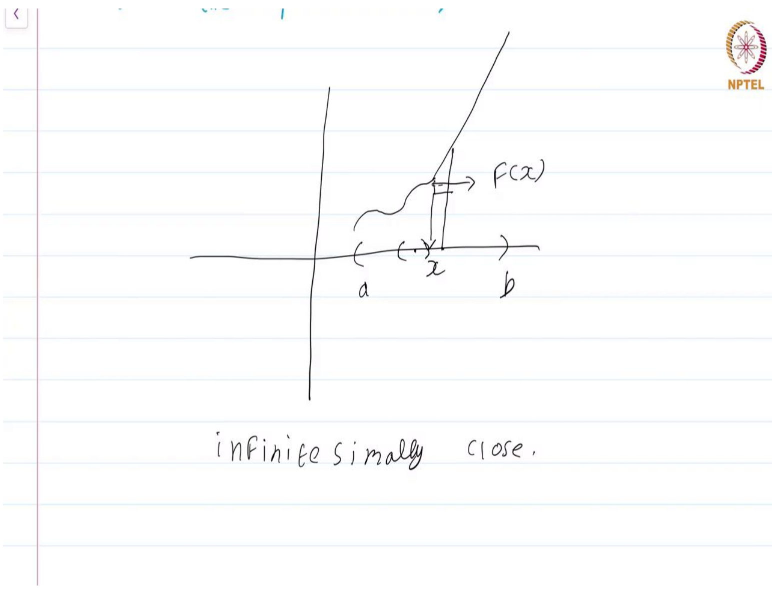
So, we take a function $F:(a,b)\longrightarrow\mathbb{R}$. We fix a point x in that interval. We want to talk about whether F is continuous at this point x.

We say F is continuous at x if for any set $A\subset(a,b)$ that is "close" to x, the set F(A) is "close" to the point F(x).

So, why this is an informal definition is because we do not really know what the term and codes close really means. All this is saying is, for a function to be continuous at a point if you take a set that is closed to that point, the image of that point and the image of that set should also be closed to each other.

So, this neatly captures our intuition about continuous functions. Now, let us try to make this notion of close precise. For that let me first draw a graph of a function view whose continuity we are trying to analyze.

(Refer Slide Time: 03:14)



So, let me draw the graph of a function; let us say this is a, this is b, let us say this is a function and this is the point x at which I want to see whether the function is continuous. Let us say what happens is, as soon as you touch x; the function sort of zooms, like the stock price of the software zoom for the past few months, let us say it just zooms.

Now, if you think about it; if I am taking a set that is close to the point x, surely you will accept that if I take. So, x is actually here to be ultra-precise; x is not somewhere in the air, x is here this is the point F(x), this is the point F(x). Now, if I were to take a set which is like this sort of an open interval to the left of x; then if I choose a point here, that is very close to x and the value of that point is also very close to F(x).

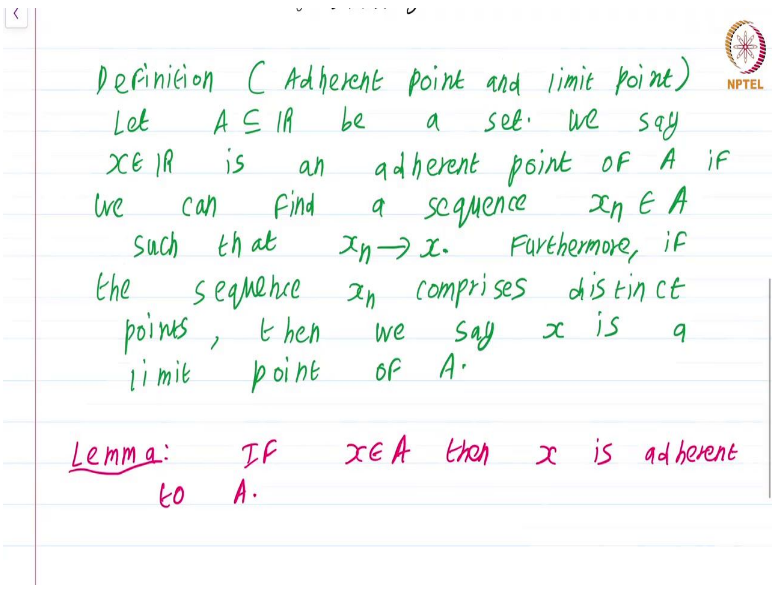
But if I choose a symmetric point on the other side that sort of goes further away. Because this function is sort of just zooming; just one second, let me just not draw that curve in such a manner as to mislead you into thinking that it is not a function. So, what we are really interested in is, not just what happens to F(y) when y is near x; but in actuality what we are interested in is, what happens to F(y) when y is infinitesimally close to x.

So, we are not just interested in formalizing the notion of close; we are interested in formalizing what is the meaning of infinitesimally close. This just seems to have made our problem even more difficult; at least the notion of close has some tangible relation to reality, whereas this infinitesimal word seems even more complicated.

If you recall, in high school a ton of manipulations algebraic manipulations in calculus was justified by saying this quantities are infinitesimally small and therefore, can be neglected.

Of course, due to the standards of the course that we are in; we have a high standard of rigor, we cannot do such things. So, we have to make precise this notion of infinitesimally close. Now, this is where this notion of closed sets come into the picture. So, let us make the first definition in this chapter on topology.

(Refer Slide Time: 06:27)



Definition (Adherent point and limit point) The word adherent means, sticks to the set that is going to capture the meaning of a point being infinitesimally close to a set. And the limit point is a more refined notion of adherent point, which is very well studied and used in analysis.

So, the definition is as follows.

Let $A \subset \mathbb{R}$ be a set. Let or rather we say $x \in \mathbb{R}$ is an adherent point; again adherent means, sticks to the set, is an adherent point of A, if we can find a sequence x_n, all of the elements of x n have to come from A, such that x_n converges to x. Furthermore, if the sequence x_n comprises distinct points; that means there is no repetition in the sequence, then we say x is a limit point of A.

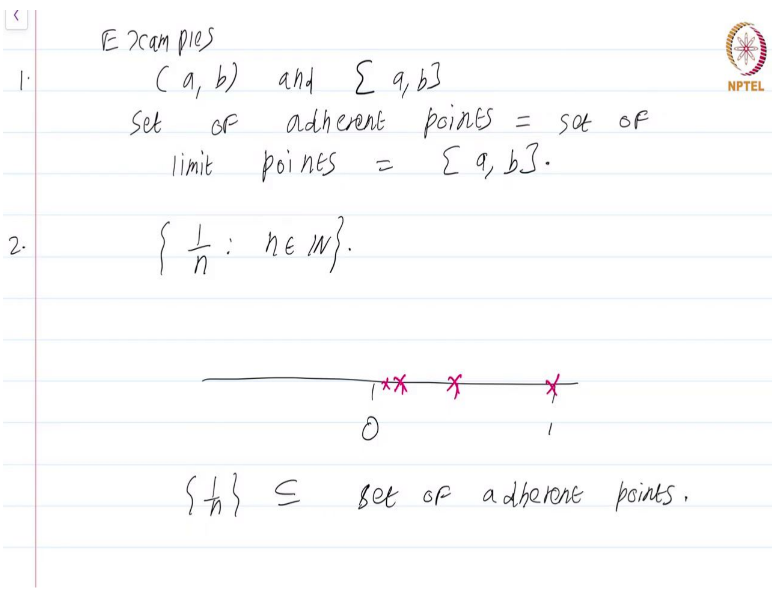
So, at the outset it should be obvious that a limit point is automatically an adherent point. So, this adherent point captures the idea that a point x is "infinitesimally close to the point to the set A". So, what we are saying is, the point x is adherent to A if you can find elements of a that converge to x.

If you think about the definition of x_n converging to x; what this is really saying is, no matter how close to x_i get, there is always a point in the set A that is closer to x. That is the only way by which there can be a sequence x_n converging to x. Now, the first lemma which is so obvious that I am going to leave it to you.

Lemma: If $x \in A$ then x is adherent to A.

Well, this is utterly obvious; if you have an element that is already in the set, it is infinitesimally close to that set. In fact, it's distance from the set is in some sense 0, right. So, this is such an obvious lemma that I am not even going to bother proving it. Now, let me give some examples to illustrate that limit points and adherent points need not always be the same.

(Refer Slide Time: 10:28)



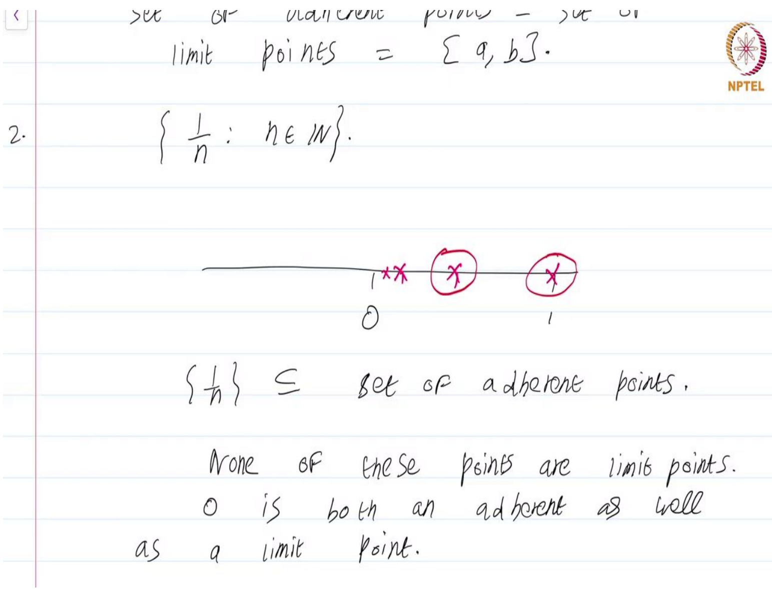
So, examples; first let us consider our favorite open intervals (a,b) and closed interval [a, b]. Now, the set of adherent points in both scenarios is equal to the set of limit points and is both are equal to the closed set [a, b].

Now, from the previous lemma, it is clear that every point in the set (a, b) will be an adherent point of both an open interval (a, b) and closed interval [a, b]. Showing that it is a limit point is also not that hard; you have to construct a sequence that converges, that is also utterly easy for the end points, it is not at all difficult to construct sequences that converge to the end points also.So, in this case, the set of adherent points and the set of limit points seem to be coinciding.

Now, let us take a drastic example; let us take the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. Now, let us plot this set; if this is 0 and this is 1, you have this point, you have one third which is somewhere here and so on, you have points getting closer and closer to 0.

Now, observe that by the fact that every point of the set is automatically an adherent point, it automatically makes $\frac{1}{n}$ a subset of a set of adherent points.

(Refer Slide Time: 12:35)

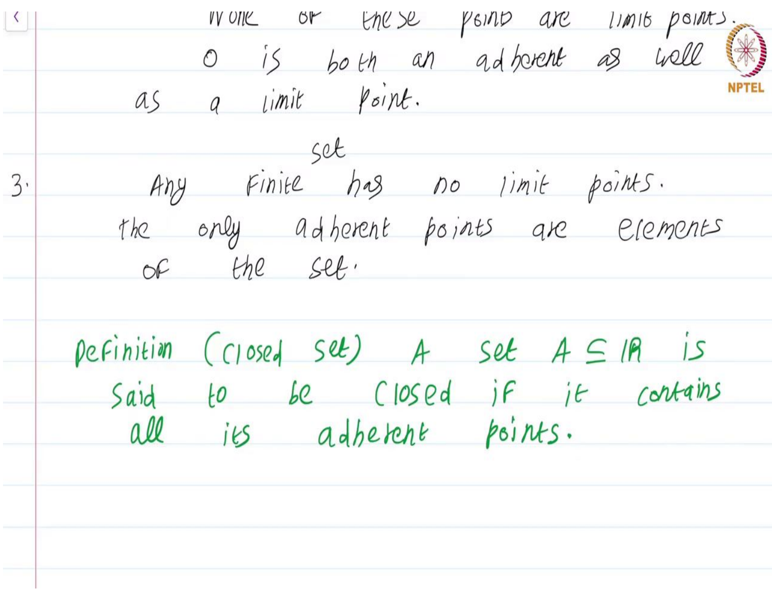


Now, it should be clear to you that, if I take any one of these red points; the only sequence in the $\{\frac{1}{n} : n \in \mathbb{N}\}$ that converges to this point is a sequence that is eventually all once. The same is true for this point which is half; the only sequence in the set $\{\frac{1}{n}\}$ that converges to half is a sequence that is eventually just half, half, half, half repetitions of half.

So, none of these points are limit points. So, here it seems like the set of adherent points and limit points are not even going to have a common point. But wait a second; what about our element 0, which is what $\frac{1}{n}$ is converging to. By the definition of both adherent and limit point, 0 is both an adherent as well as a limit point.

So, the set $\frac{1}{n}$ is sort of getting infinitesimally close to the point 0, right. So, here the set of limit points is just one element, the element 0; whereas the set of adherent points is $\frac{1}{n}$ along with 0. So, these two notions need not always coincide.

(Refer Slide Time: 14:16)

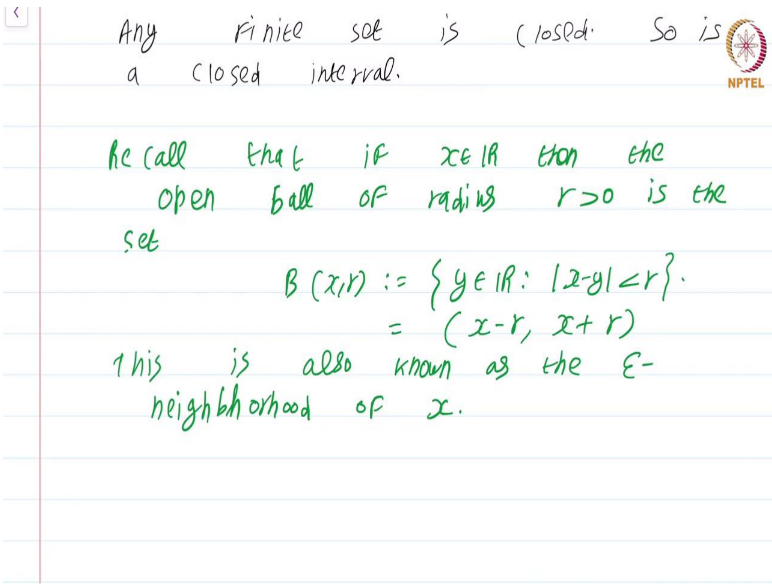


As a drastic example, any finite set has no limit points. And the only adherent points are elements of the set, excellent. So, this is another drastic example. So, we have somehow captured the fact that a set is getting really close to a point. Now, I come to the central definition of topology; in fact many will say this is the second central definition, but it is really the same.

Definition (Closed set): A set $A \subset \mathbb{R}$ is said to be closed if it contains all its adherent points.

That means the set contains any point that gets infinitesimally close to that set. So, examples of closed sets we already see that, any finite set is closed. So is a closed interval.

(Refer Slide Time: 16:07)



Before I proceed further with the study of closed sets; let us define what is called the dual notion of closed sets, which is the notion of an open set. For that I need to define what it means for a point to be an interior point. The notion of an adherent point captures that a particular element is infinitesimally close to the set A. The notion of an interior point captures the fact that the set at the point is fully inside the set A.

So, to make this precise, I need to recall that;

If $x \in \mathbb{R}$; then the open ball of radius r > 0 is the set $B(x,r) := \{y \in \mathbb{R} : |x-y| < r\} = (x-r, x+r)$. This is also known as the ϵ-neighborhood of x.

(Refer Slide Time: 18:17)

This is also known as the ε-neighborhood of x.

Definition (interior point). Let A be a set. A point $x \in A$ is said to be an interior point if $B(x,r) \subseteq A$ for some $r > 0$.

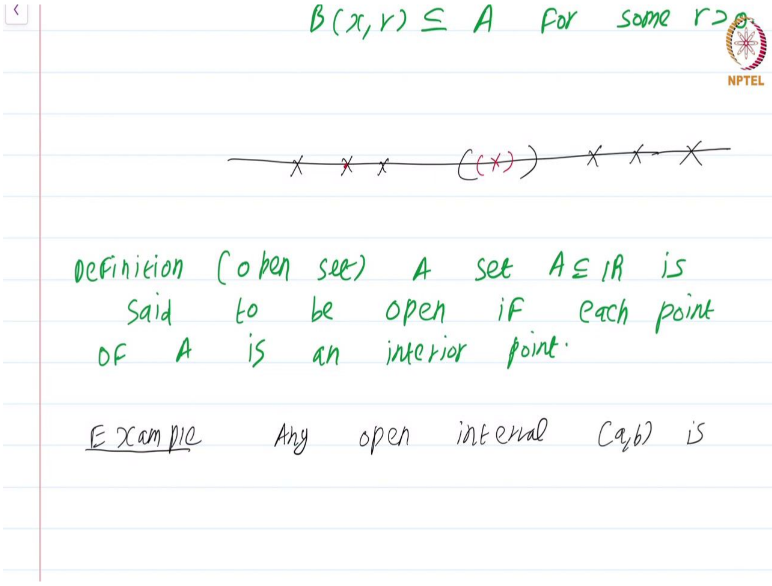
Now, we can define the notion of an interior point and open set.

Let A be a set. A point $x \in A$ is said to be an interior point; if $B(x, r) \subset A$ for some r > 0. So, the best way to illustrate this is of course by a picture; you have a set A on the real line, let us say there are these finitely many points and there is this open interval, there are some more finitely many points.

If you choose a point inside here, then I can of course find an open interval around this point which is fully contained in the set. Whereas, if I am here, I cannot do the same. So, it is sort of capturing the fact that a point is fully inside the set, not only is the point there in the set; but some neighborhood of the point is also there in the set, then we say that it is an interior point.

So, the definition is, captures our intuition of what it means for a point to be fully inside a set quite nicely.

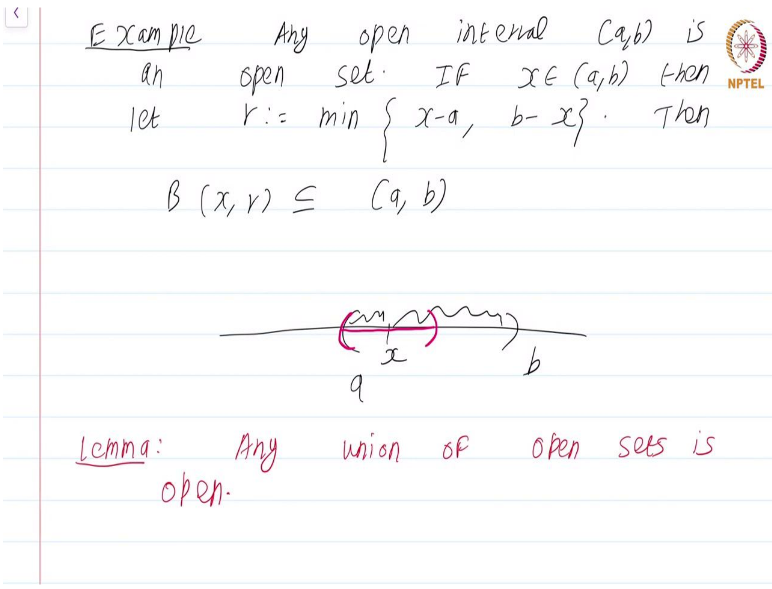
(Refer Slide Time: 20:09)



Definition (Open set) A set $A \subset \mathbb{R}$ is said to be open , if each point of A is an interior point. Let us give one example at least.

Example: Any open interval (a, b) is an open set.

(Refer Slide Time: 21:04)



Well, how do you show this? Well, it is rather easy. If $x \in (a,b)$; then let $r := min\{x-a, b-x\}$ then $B(x,r) \subset (a,b)$.

So, what this says pictorially is if I choose the open interval (a, b), choose the point x here; I am looking at the minimum of these two distances and setting it to be the radius r. Then all I am saying is, when you do this r radius ball around the point x that is fully contained in (a, b) that is the way r was chosen.

Now, it should be immediately clear to you that this lemma is true; I would not even prove this lemma, it is that easy.

Any union of open sets is open.

It does not matter; it could be that you take infinitely many open sets also and take its union, it will still be open.

Why is this the case? Well, if you take a point x in this union, it has to be in one of the things which you are taking a union and that set is open; and that means, every point of that set is an interior point and now you can finish this argument.

So, now I have defined the basic concepts of open and closed sets; let us proceed with our development from the next module.

This is a course on Real Analysis and you have just watched the module on Open and Closed Sets.