Real Analysis - I

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Lecture – 13.1
The Role of Topology in Real Analysis

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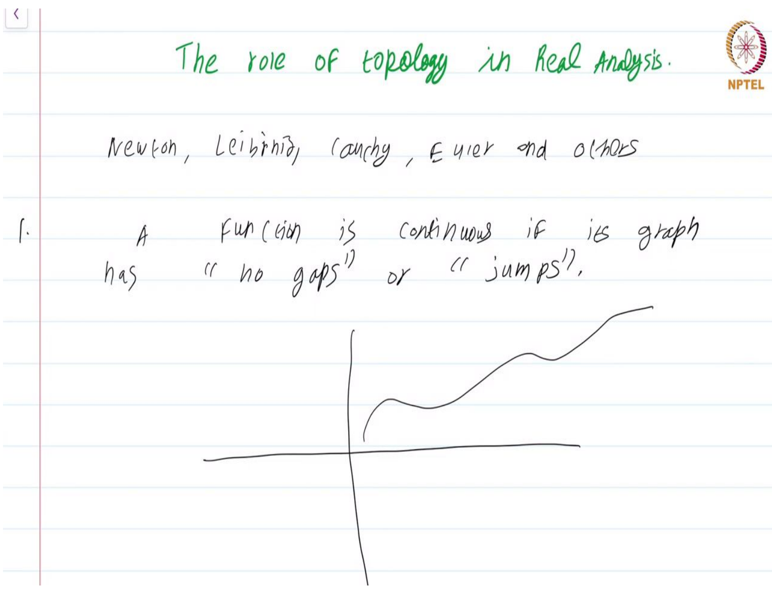

For the next three weeks, we shall be discussing the interesting topic of topology and continuous functions. Now, it is possible to do the rest of this course without using the word topology or tall. There is no need to invoke anything from topology, other than the basic facts about open intervals.

But nevertheless, this is like learning the subject with one eye closed. I want to introduce topology; because not only does it provide us with a convenient framework to study continuous functions, it is also the right framework to study continuous functions from.

In fact, one of the earliest books in topology by Kelley was originally conceived as what every young analyst should know; but later the publishers convinced him not to have that title, it is now just known as general topology by Kelley. So, topology is no doubt extremely essential in analysis; but I want to motivate why it is needed even at such an elementary stage to understand continuous functions fully.

To do that, let us first try to understand what a possible definition of continuity could be. Now, here is a philosophical issue; the idea of a continuous function exists purely as an intuitive idea. Our aim is to make this intuitive idea mathematically precise. Now, how do you do that? Well, you contemplate the intuitive idea and try to write a rigorous definition which tries to capture that intuitive idea.

So, before we even begin trying to write down a definition of what a continuous function is; we must first clarify what it is that we have in our mind. If you remember when we defined sets; we could not actually define them precisely, we have to resort to an axiomatic characterization.

And you notice that, once we had axiomatically characterized the sets, there are number of properties of sets which were not anticipated by our intuition; indeed the existence of sets that are uncountable, that was not something that we had anticipated when we set out to create these axioms of set theory.

Now, we are going to define continuous functions precisely not just characterize them axiomatically and it will turn out that they have a lot of properties that we desire; but some non-intuitive properties also. So, let us first try to write down what a continuous function should be intuitively.

Now, several mathematicians Newton, Leibniz, Euler, Cauchy have given several various intuitions as to what a continuous function is. Let us write down some of them; I am not going to bother mentioning who gave what, because I myself have not read the original works.

So, Newton, Leibniz, Cauchy, Euler and others have given various intuitions. Now, what are they; let us list them. One, a function is continuous, if its graph has no gaps or jumps. This is one idea of what a continuous function could be. Well, let us see an example of a function that is indeed continuous; I will just draw something which is nice and continuous.

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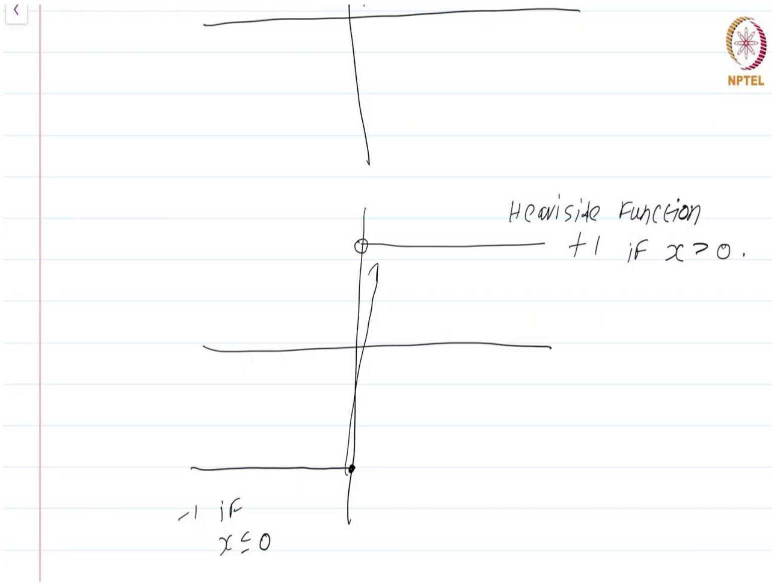

Now, let us draw something which is obviously not continuous; this is known as the heavy side function, it is very important in physics and engineering. This function is just $-1\ if\ x \leq 0$, and +1 if x > 0. I have put this dot and circle just to sort of graphically illustrate that at the point 0 the value of the function is -1 and not 1;.

So, this is known as the heavy side function; it is very important in physics and engineering. This function jumps, it jumps at the point 0; clearly we would not want this function to be a continuous function, this is a jump or a gap whatever you want to call it.

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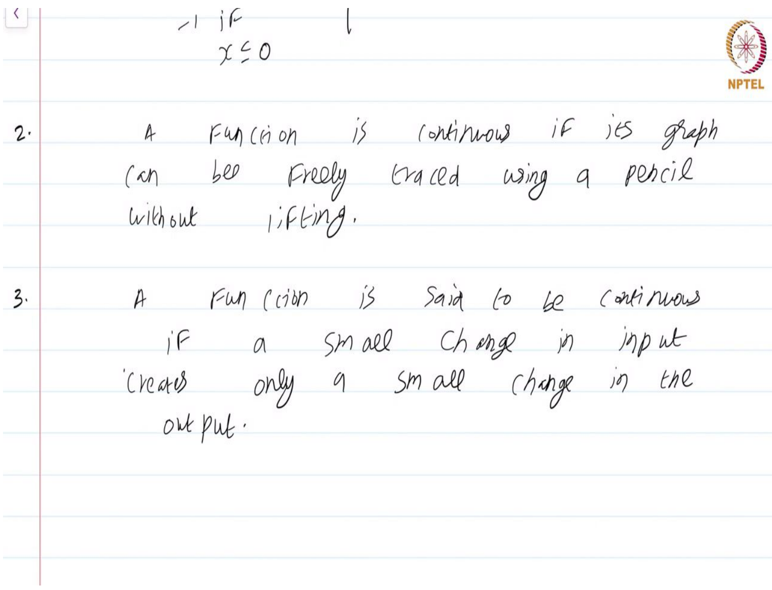

That is one intuition. Another intuition of what a function is.

A function is continuous; if its graph can be freely traced using a pencil without lifting it. Now, this should remind you of several puzzles that you might have done when you are a kid; you are given a set of dots, you have to join all the dots together and get back to the original point without lifting up your pencil. So, this is one more idea of what a continuous function should be.

Yet another version, a function is said to be continuous if a small change in input creates only a small change in the output. So, we can view the function as an input, output machine; you feed in something, you get back something out. If you slightly perturb your input, the output should not change too much; this is yet another definition or rather an intuition about continuity.

So, now, we have three sort of plausible intuitions of what a continuous function is supposed to be. And as stated, none of them really seem to be mathematically precise; the one that seems to be most mathematically precise is the third one. The other two are really difficult: how do you capture a gap, how do you capture it, a function can be freely drawn using a pencil without lifting it and what not.

Well, let us try to recall back, way back in the very first lecture; I talked about the intermediate value theorem being non-obvious. Recall what the intermediate value theorem sort of says that, if you have a function that is continuous and it takes two values; then it must take all values in between also.

This sort of seems to be capturing the fact that, there is no drastic change or there is a continuous change or you could be able to trace it without lifting your pencil and all that. It seems like the intermediate value theorem captures the notion of continuity. Well, let us just write down a mock definition.

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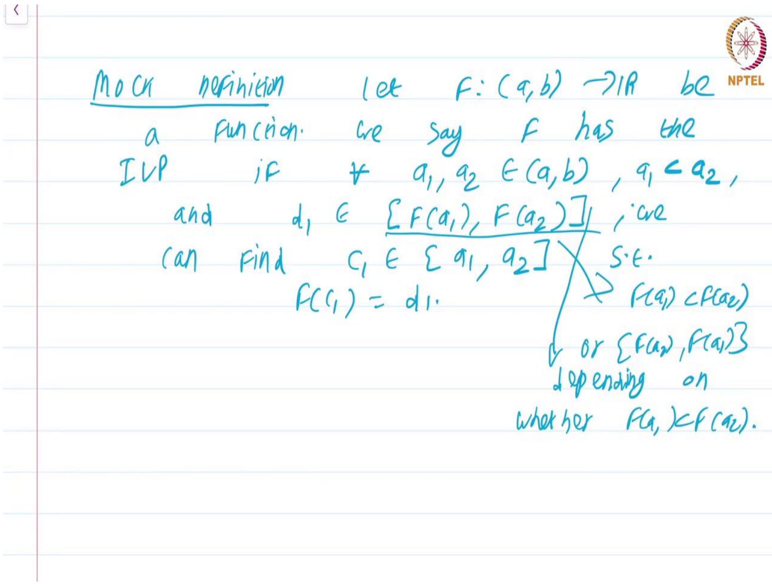

This is not the real deal.

Let $F:(a,b) \longrightarrow \mathbb{R}$ be a function. We say F has the intermediate value property if $\forall a_1, a_2 \in (a,b),\ a_1 < a_2$ and $d_1 \in [F(a_1), F(a_2)]$, we can find $c_1 \in [a_1, a_2]$ such that $F(c_1) = d_1$.

Here I am assuming $F(a_1) < F(a_2)$; otherwise this closed interval does not make sense. If $F(a_1) < F(a_2)$, then I will replace this interval by $[F(a_2), F(a_1)]$.

So, let me just or $[F(a_2), F(a_1)]$ depending on whether $F(a_1) < F(a_2)$. So, what this is saying is, if two values $F(a_1)$ and $F(a_2)$ are attained at the points a_1 and a_2; then all the intermediate values will be attained by some points within a_1 and a_2.

This sort of seems to be capturing the fact that the graph can be traced continuously by the function. Well, this is a definition that is straightforward and seems to capture our notion of continuity; but it is wrong. What do I mean by the definition is wrong?

Well, I can produce a function that will certainly be continuous and will satisfy the intermediate value property and if you accept this definition as a definition of continuous, it will be continuous; but we do not want that function to be continuous, that will also be clear to you. Well, let us define that function.

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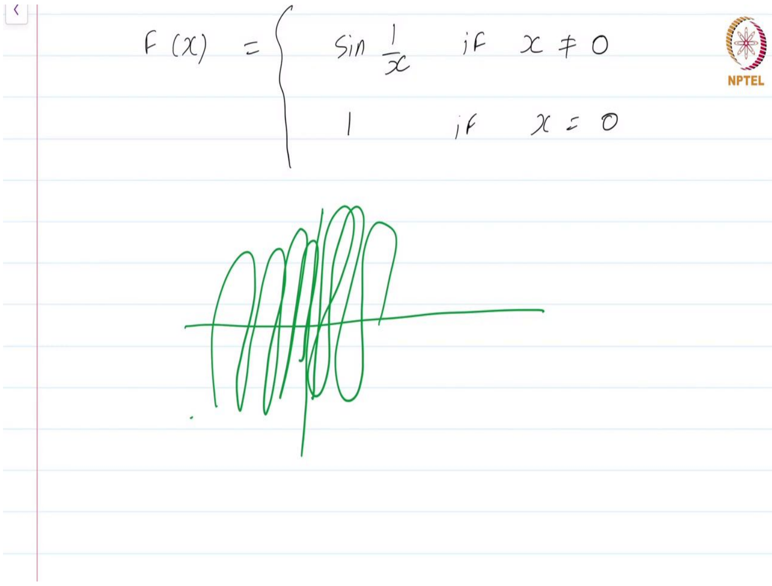

This function $F(x) := \begin{cases} \sin\frac{1}{x}, \ if\ x \neq 0 \\ 1, \ if\ x = 0 \end{cases}$.

I am defining a function in two pieces; its $\sin\frac{1}{x}$, if $x \neq 0$, 1 if $x = 0$. Now let me attempt to draw this function, it is insanely difficult; what happens is as x gets smaller and smaller, 1 by x becomes larger and larger and what happens is as you approach 0, this function starts oscillating in between minus 1 and 1 very rapidly.

So, let me attempt to draw this, do not know how successful I will be, something like this; it was very badly drawn, but you get the idea. So, the same thing will happen in the negative axis; the function will start oscillating widely as you start approaching 0.

Now, what is happening is, even though this function seems to be continuous everywhere except possibly at the origin. it is oscillating too rapidly near the origin, but it is still oscillating in a manner such that the intermediate value theorem will be true for this function. This function will have the intermediate value property.

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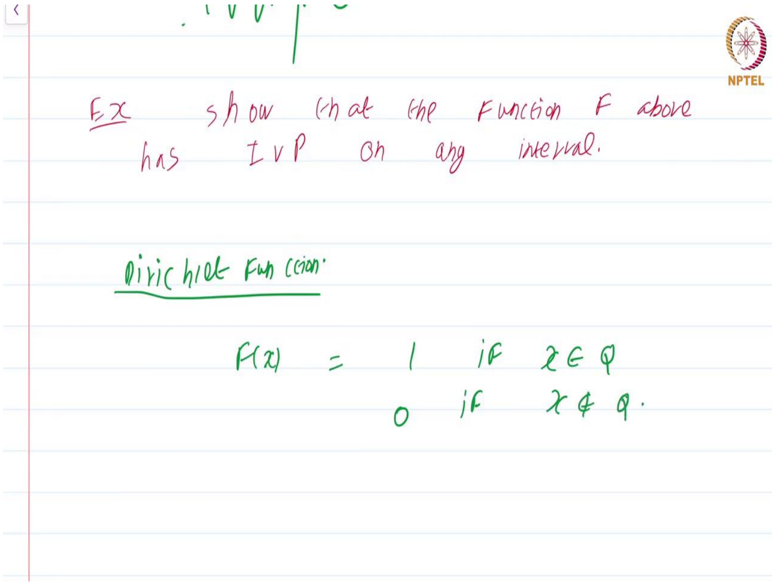

So, first Exercise for you. Show that the function F above has the intermediate value theorem property on any interval.

So, what is happening is, it is possible for a function to oscillate very rapidly; but still have the intermediate value property, that is sort of intuitively clear. Intermediate value property just says that, all value should be taken in between; this function does that, but it does that so fast and rapidly that no sane person would want to call this function a continuous function.

So, the candidate definition using intermediate value property fails. So, we will have to try and capture the definition of continuity using the fact, using the intuition that a small change in input creates only a small change in the output.

Well, to try to understand how we capture this; it might be good to consider some functions that are extremely weird that can challenge our intuition, like the one that we just saw.

Let me give an example by Dirichlet. This is the Dirichlet function; this is defined as follows,

$$F(x) := \begin{cases} 1 \; if \; x \in \mathbb{Q}, \\ 0, \; if \; x \notin \mathbb{Q} \end{cases}.$$

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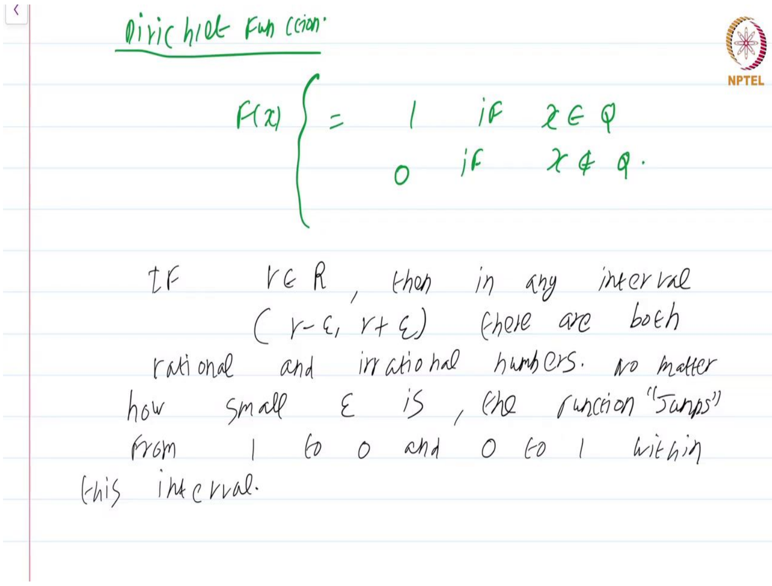

Again this function is very very difficult to plot; because rational numbers are everywhere on the real line. We already saw that between any two real numbers, there is always a rational number. So, what happens is, there will be plenty of jumps in this function as it moves from rational to irrational and irrational to rational, it will keep jumping from 1 and 0.

So, if $r \in \mathbb{R}$, then in any interval $(r-\epsilon, r+\epsilon)$, there are both rational and irrational numbers. This is something that we proved way back when we studied the basic properties of real numbers. So, there are both rational numbers and irrational numbers. That means no matter how small, ϵ is the function jumps from 1 to 0 and 0 to 1 within this interval.

In fact, there will be infinitely many jumps, in fact uncountably many jumps no matter how small this interval is. So, this function, it does not oscillate rapidly, it sort of jumps rapidly. So, this also somehow we have to capture that continuity should disallow such functions.

So, we should expect this function to be continuous nowhere on the real line, that is what our intuition suggests. Now, how do we make this intuition somewhat more precise? Well, I just said that no matter what interval you take you will always get both the rational number and an irrational number.

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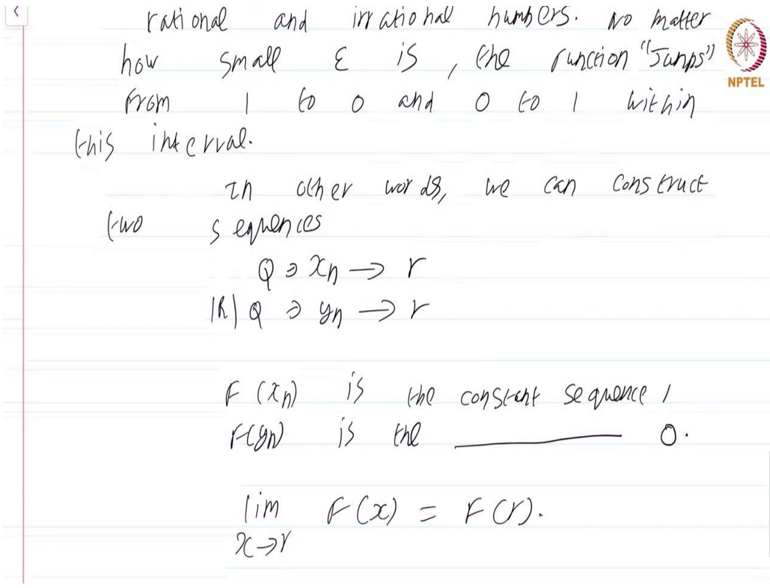

In other words, we can construct two sequences, x_n which are all elements of $\mathbb{Q}$ that converge to r and y_n which are all elements of $\mathbb{R} \setminus \mathbb{Q}$. That means irrational numbers that converge to r.

Now, if you look at the sequence $F(x_n)$, you will get constant sequence 1, whereas $F(y_n)$ is the constant sequence 0. So, what is happening , we have found two sequences that converge to the point r, such that $F(x_n)$ and $F(y_n)$ do not converge to the same value.

So, this sort of suggests that we must explore continuity using sequences. And not only should we explore continuity using sequences; we must not make the mistake of considering just one sequence converging to the point r. For instance, if r were a rational number; then we already know that F(r) is going to be 1.

If I had just considered the rational sequence x_n converging to r, I would get $F(x_n)$ also converges to 1, which is F(r); which would sort of mislead me into believing that this function F is continuous at this point r, but that is not the case.

We must not just consider one sequence converging to r; we must consider the totality of all sequences converging to r and then stipulate that F of all those sequences should also converge to the same element. Then it sort of seems like F will be continuous at that point.

Well, we still need to show that if we take this as the definition of continuity. It must capture our intuition that a small change in the input produces only a small change in the output.

We will explore this in the future, that is what this entire three weeks are going to be about; but for the time being I hope, I have convinced you that there is some sort of relationship between continuity and sequences. This idea that a small change in the input produces only a small change in the output can be captured by saying that $\lim_{x \to r} F(x) = F(r)$, if it is continuous. If it happens that for every point r in the interval, $\lim_{x \to r} F(x) = F(r)$.

This is what we have learnt in high school as the definition of continuity. Now, let me try to explain what topology is and the relationship between topology and what we are studying? Topology abstractly is the study of space.

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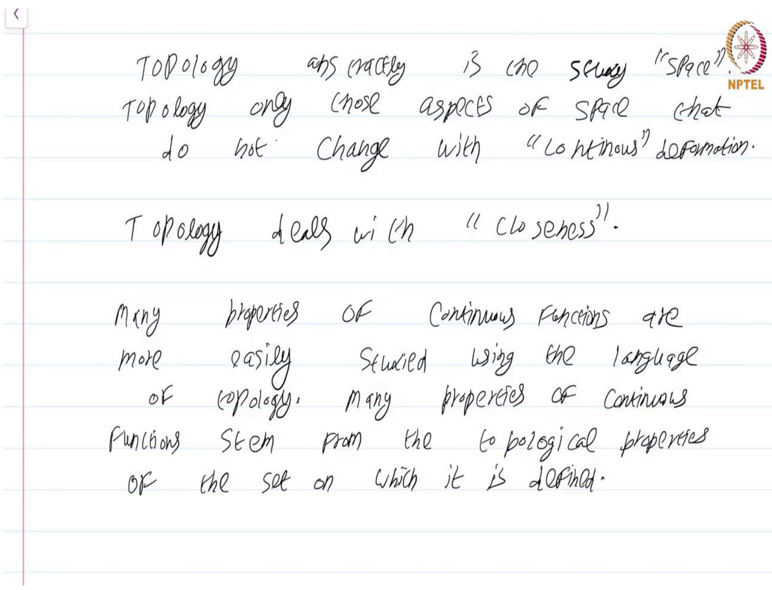

This is a very very vague definition. More precisely it studies only some aspects of space. Topology studies only those aspects of space that do not change with continuous deformations. So, sort of continuity is built right into the very definition of what topology studies.

Now, this is an abstract intuition behind what topology is, I will not elaborate on this further; but for the purposes of this course, I can just say, topology deals with closeness, which is more relevant to our course.

Now, why closeness has anything to do with the study of space is something that I do not want to get into. So, topology deals with closeness and closeness is what is needed; the idea of what it means for things to be close to each other is needed to capture our intuition of continuity and limits, so topology naturally comes in.

Not only that, many many properties of continuous functions are more easily studied using the language of topology. For instance, we will define the notion of limits using what is known as the $\epsilon - \delta$ definition of limits. Similar to what we have already seen for sequences, where there was an ϵ and a N. Similarly we will have an ϵ and δ.

But the language of topology allows us to put all that under the carpet and talk in more intuitive language using open sets and neighborhoods and so on and it will allow us to have a nice and reasonably good language to talk about continuous functions. Not only that, many properties of continuous functions stem from the topological properties; the topological properties of the set on which it is defined.

One of them is the intermediate value property that depends on the fact that closed and open intervals are connected sets, they have only one piece. Then the second fact is, continuous functions attain both their maxima and minima. This is due to the fact that continuous functions attain maxima and minima on closed intervals, this stems from the fact that closed intervals are in fact what are known as compact.

So, many many properties of continuous functions actually stem from the fact, from the underlying features of the sets on which they are defined. So, for this reason, we need to study the various topological properties of the real line, which we shall be embarking on this week and partially next week. Then we shall study continuous functions and the interplay between continuity and topology.

This is a course on real analysis and you have just watched the module on the role of topology in a real analysis.