Real Analysis - I
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Lecture – 12.5
The Cauchy Product

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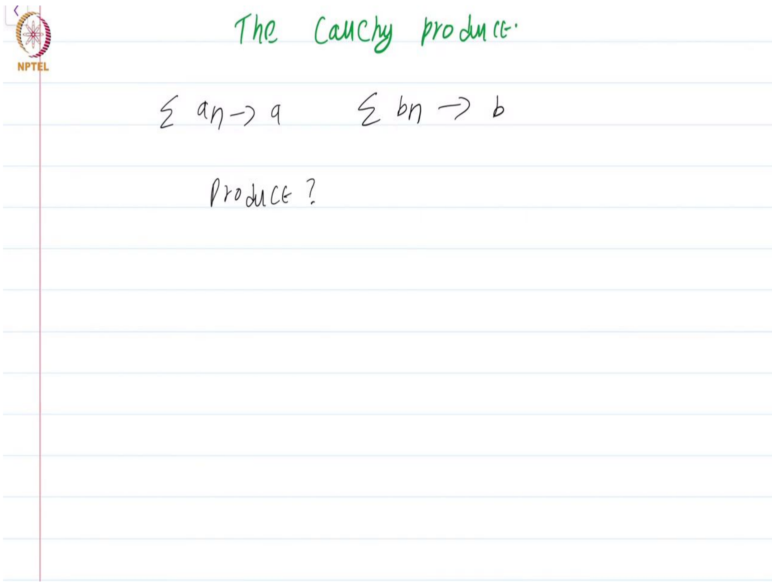

Suppose we are given two series, $\sum a_n$, $\sum b_n$ and $\sum a_n \to a$ and $\sum b_n \to b$. Then does it make sense to take the product .Is there a natural product on this.

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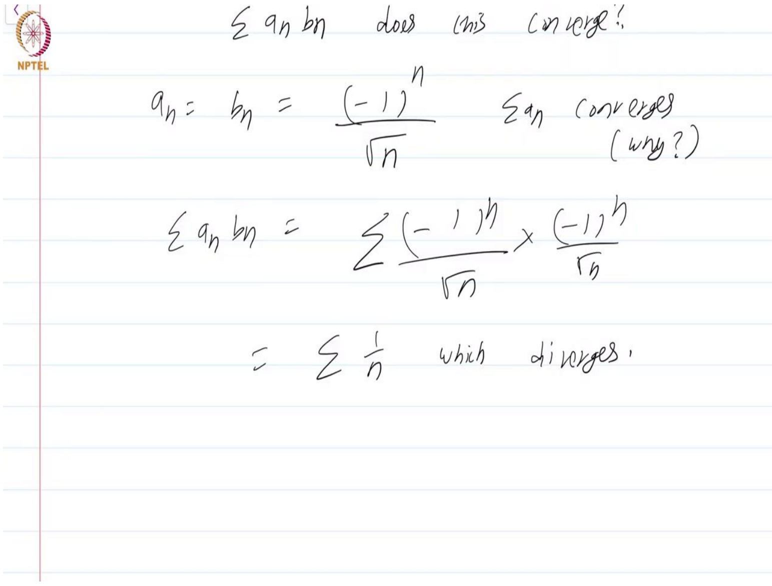

Well naively you would think that we can consider the product $\sum a_n b_n$. Does this always converge? Well let us take an example: suppose I take $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Then, $\sum a_n$ converges. Think about why this is the case .In fact, we have got a precise test that sort of tells you why this converges.

Then $\sum a_n b_n$ is nothing, but $\sum \frac{(-1)^n}{\sqrt{n}} \times \frac{(-1)^n}{\sqrt{n}}$ which is just $\sum \frac{1}{n}$ which diverges. So, it's not always the case that, if you take a series of the form $\sum a_n$ and another series of the form $\sum b_n$ and naively take the product $\sum a_n b_n$ that need not converge. Let us try to take a more refined product.

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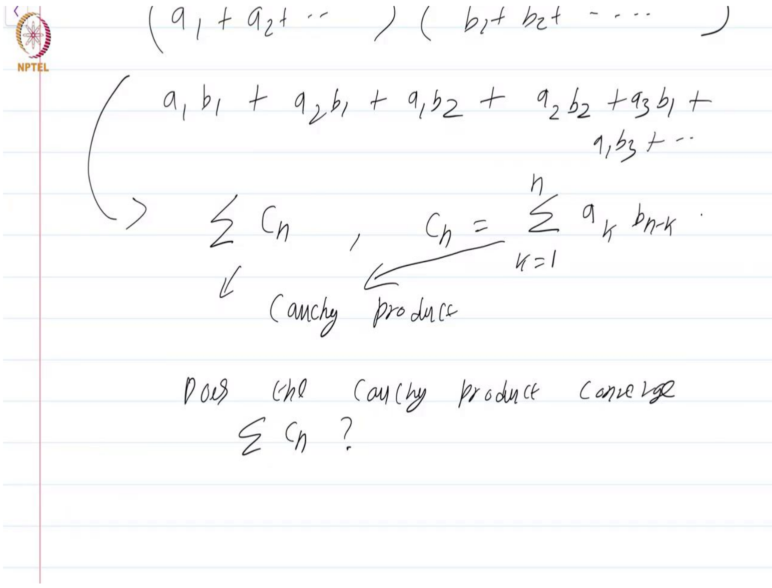

What do we do in the following? Observe that the terms can be written like this $(a_1 + a_2 + ...)(b_1 + b_2 + ...)$. Then, rather than taking the series $\sum a_n b_n$, this product naturally by distributivity looks like $a_1 b_1 + a_2 b_1 + a_1 b_2 + a_2 b_2 + a_3 b_1 + a_1 b_3 + ...$.

I can write this product sort of by simply manipulating it algebraically without worrying about issues like convergence and validity, I can write it as $\sum c_n$ where c_n is $\sum_{k=1}^{n} a_k b_{n-k}$.

That means I am grouping together all the terms from a_i and b_j such that i+j=n, that is how I am grouping the terms together this produces, this c_n. So, this each c_n is actually just $\sum_{k=1}^{n} a_k b_{n-k}$. So, this is a different product than this naive product $\sum a_n b_n$, this is called the Cauchy product of the two series.

Now, the question arises: does the Cauchy product converge? In other words what is $\sum c_n$? Now, I am going to prove a very simple result, this is not the most general result on the slides, but it's more than sufficient for our purposes.

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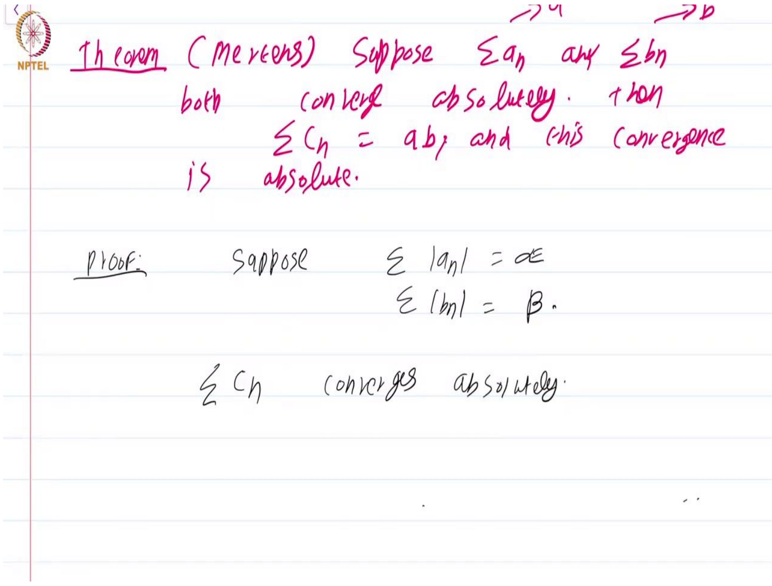

Theorem, I believe this is due to Mertens, but I think what Mertens proved is a more general result, what I am about to prove is much simpler. It is probably known much before Mertens.

Suppose $\sum a_n$ and $\sum b_n$ both converge absolutely, say $\sum a_n \to a$ $\sum b_n \to b$ both converge absolutely. Then, $\sum c_n$ is just as you can guess a b and this convergence is absolute.

Let us see a proof and the proof is not very hard because, I am assuming both series converge absolutely.

Suppose $\sum |a_n| \alpha$ and $\sum |b_n| = \beta$. The aim is to show that $\sum c_n$ converges absolutely, that is the first claim in this theorem.

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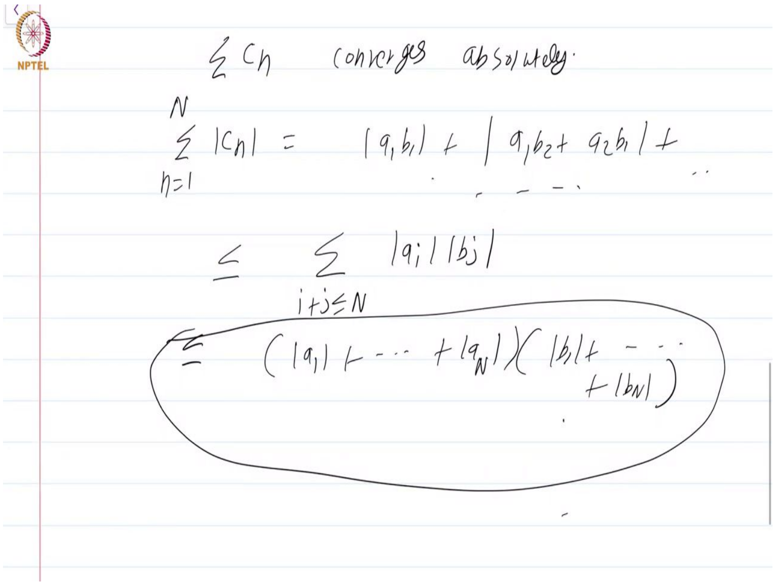

Now, how we are going to show that $\sum c_n$ converges absolutely. Well observe the following. Look at $\sum_{n=1}^{N} |c_n|$. Now, this will consist of terms that look like this $|a_1 b_1| + |a_1 b_2 + a_2 b_1| + \ldots\ldots$. This is certainly going to be less than or equal to $\sum_{i+j \leq N} |a_j||b_j|$.

I have just applied the triangle inequality to the various terms in the previous sentence. Now, here is the catch: this is in fact, less than or equal to $(|a_1| + +|a_N|)(|b_1| + . + |b_N|)$. Notice that all the terms in this expression after you have expanded it out using distributivity, all the terms here will be of the form $|a_i b_j|$, but with the possibility that i + j could exceed n .

So, I must be precise I put a N that makes no sense it should be I mean I put a 'n' it should be N . So, every term that occurs here is of the form $a_i b_j$, but i + j could be greater than or equal to N, should be greater than N. Therefore, you have this inequality that $\sum_{i+j \leq N} |a_j||b_j|$ is less than or equal to this product. And this is certainly less than or equal to $\alpha\beta$.

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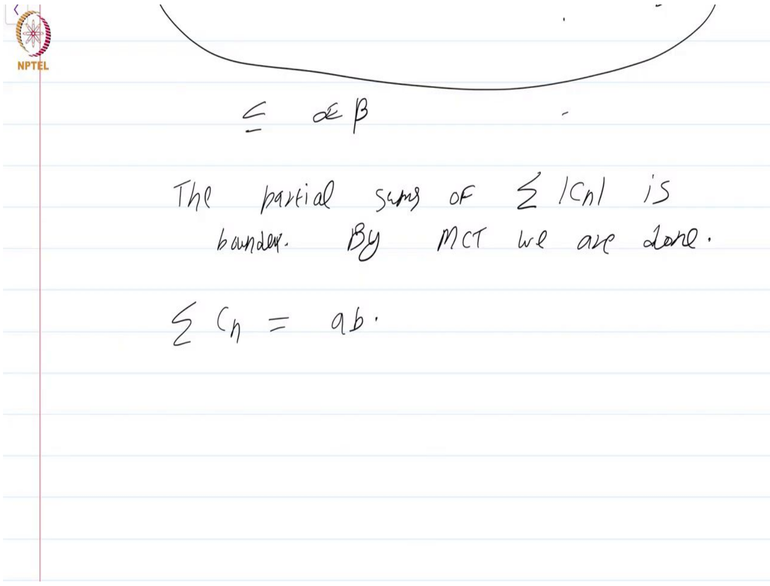

Because, that is what $\sum |a_i|$ and $\sum |b_i|$ converge to . So, what this shows is that the partial sums of $\sum |c_n|$ is bounded. By the monotone convergence theorem we are done. We have shown the absolute convergence of $\sum |c_n|$. Now, let us go to the second part, where we have to show that $\sum c_n$ actually is equal to ab and here the trick is not that different.

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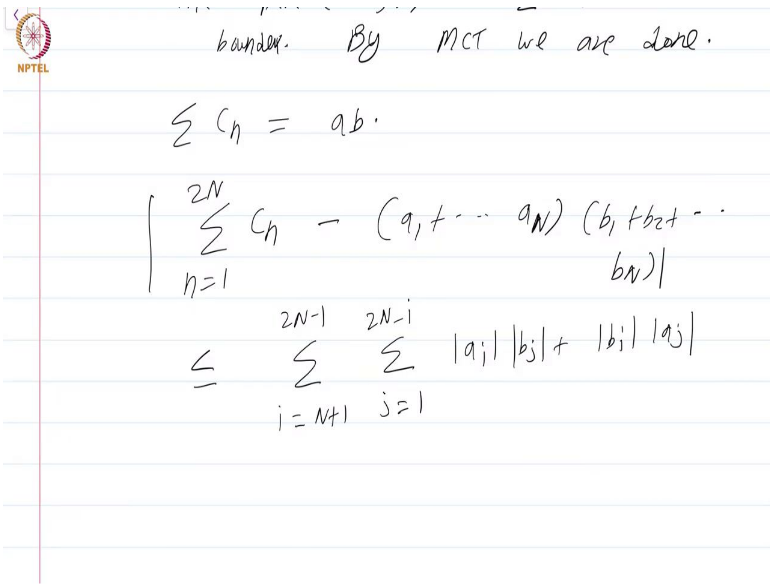

What we do is we consider $|\sum_{n=1}^{2} Nc_n - (a_1 + . + a_N)(b_1 + b_2 + ... + b_N)|$. Let us look at this difference, if you think about this difference for a couple of minutes you will notice that there will be plenty of cancellations. And the only terms that will be left behind are those of the form $a_i b_j$, where at least one of i or j is greater than 2n.

So, let us write that down we perform all these cancellations and then apply the triangle inequality and you can see in a few minutes of thought that what you will be left with is this quantity. $\sum_{i=N+1}^{2N-1} \sum_{j=1}^{2N-i} |a_i||b_j| + |b_i||a_j|$.

So, these are all the terms of the form $a_i b_j$, where one of the indices is at least N + 1 and after applying triangle inequality I have written it down as, $\sum_{i=N+1}^{2N-1} \sum_{j=1}^{2N-i} |a_i||b_j| + |b_i||a_j|$. So, if you understand this step, the rest of the proof is fairly easy. Now, what I do is every occurrence of $|b_j|$ in this first term I replace by β and every occurrence of $|a_j|$, in this second term I replace by α.

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$i = N+1 \quad j = 1$

$$\leq \sum_{i=N+1}^{2N-1} \left(|a_i| \beta + |b_i| \alpha \right)$$

$$\leq \sum_{i=N+1}^{\infty} |a_i| \beta + \sum_{i=N+1}^{\infty} \alpha |b_i|$$

tails of a convergent series.

So, I get this to be less than or equal to $\sum_{i=N+1}^{2N-1}(|a_i|\beta + |b_i\alpha|)$. Just to ensure that it is clear that the summation is over both quantities, let me just put parenthesis .

Now, simplifying again this is less than or equal to $\sum_{i=N+1}^{\infty}|a_i|\beta + \sum_{i=N+1}^{\infty}\alpha|b_i|$. Now, both of these quantities are tails of a convergent series, both quantities are tails of a convergent series.

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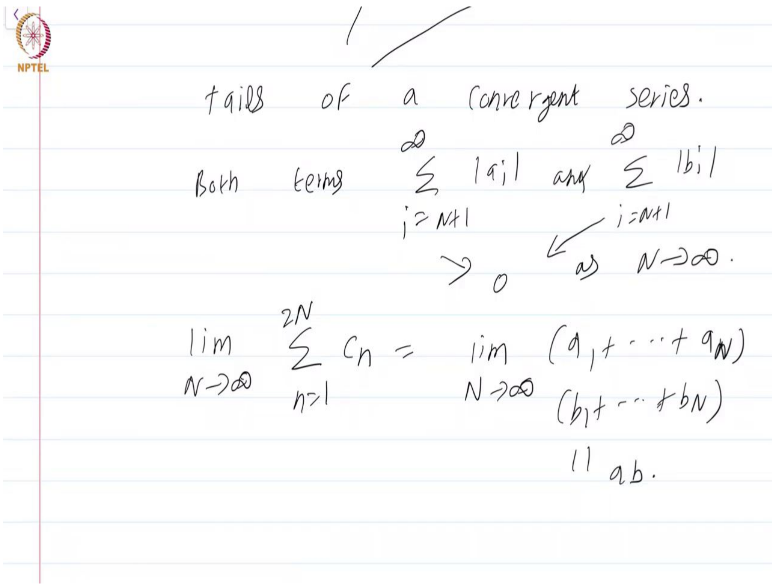

This means that both terms $\sum_{i=N+1}^{\infty}|a_i|$ and $\sum_{i=N+1}^{\infty}|b_i|$, both of these converge to 0 as $N \to \infty$, being the tails of a convergent series. This just means that $\lim_{N\to\infty}\sum_{n=1}^{2N} c_n = \lim_{N\to\infty}(a_1 + ... + a_N)(b_1 + ... + b_N)$. And this we know is equal to ab.

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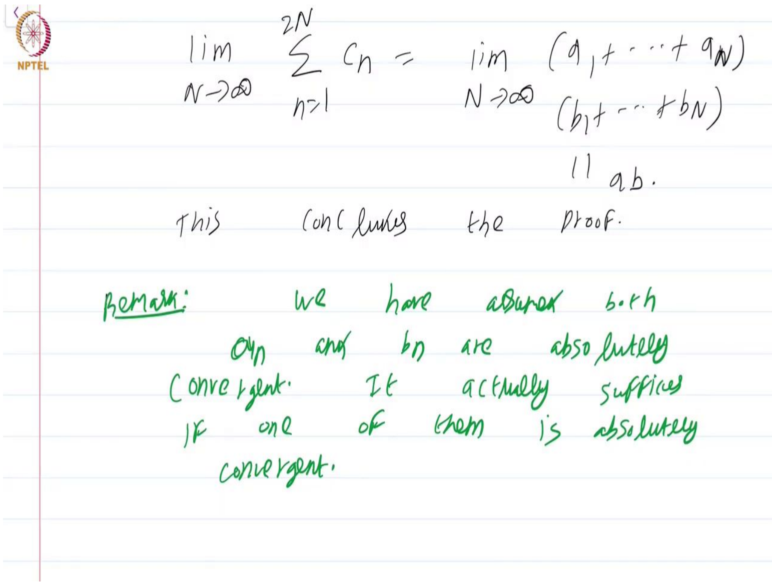

So, this concludes the proof.

So, the second part is a bit tricky, but not really difficult. Just go through the proof once or twice, look through the notes also and make sure you understand which terms cancel and what terms, we are left with and indeed. We have that $\lim_{N\to\infty} \sum_{n=1}^{2N} c_n$ is in fact equal to a b.

Now, let me just make one remark, just one remark. So, we have assumed both a_n and b_n are absolutely convergent. This is just for simplicity, it actually suffices if one of them is absolutely convergent. The other needs to be just convergent.

I am not going to prove this more general result. This result that I have stated and proved is usually sufficient for most of analysis, but it's good to know that there is a more general statement available, which you can read up on your own.

This is a course on real analysis and you have just watched the module on the Cauchy product.