Real Analysis - I

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Lecture – 12.4
Grouping Terms of an Infinite Series

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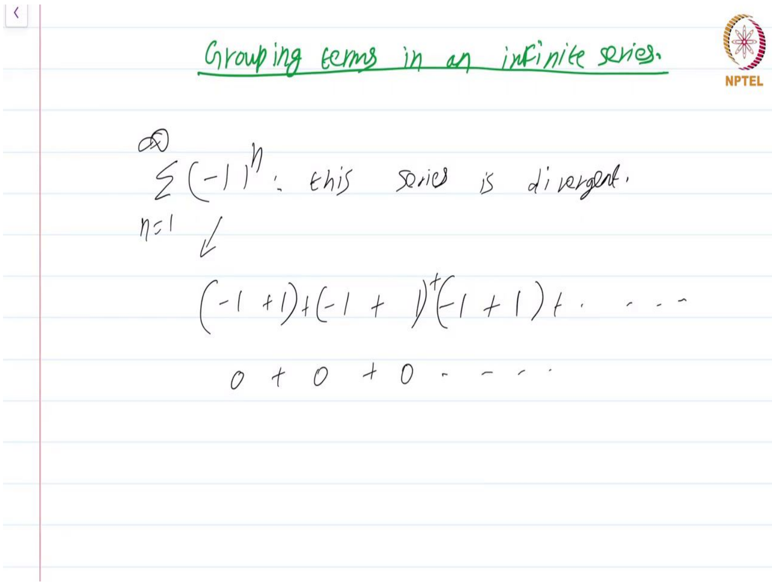

We have already seen that if a series is absolutely convergent, then rearranging the terms of the series is not going to affect its sum or its convergence behavior. Let us now see another algebraic operation that you could do to an infinite series. This operation is best illustrated by an example,

Consider the series $\sum_{n=1}^{\infty} (-1)^n$.

Now, it takes less than 30 seconds to show that this is divergent. But by grouping terms together not rearranging; note my choice of words, by grouping terms together it is possible to show that this series is convergent. Of course, such a proof is faulty, but it is a good idea to first perform this proof and see where it goes wrong.

Well, what you do is; you look at this as just (-1+1) + (-1 + 1) + (-1+1) +...... What I do is I group these terms together, I write this like this; put a + sign here group it like this and you get the general idea; I get 0+0+0+......

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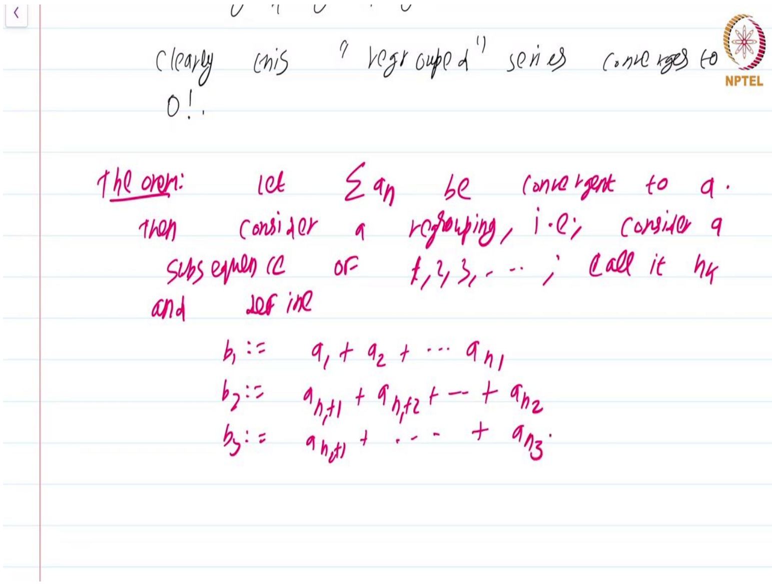

Clearly, this regrouped series converges to 0 . Why does this happen if this is weird; the original series is clearly not convergent, whereas this regroup series is clearly convergent to 0,.What is going wrong? Well, not to spoil your fun; I will just leave it to you to think about this, but let me just give you a hint. When you consider this regrouped series, you are not considering all partial sums of the original series; you are considering only some special partial sums.

So, this is analogous to there being a sequence that diverges, but having some sub sequences that converge; that is exactly what is happening. So, what I am going to show now is that if the original series is convergent; then such groupings are still going to maintain convergence behavior.

Theorem: Let the series $\sum a_n$ be convergent; convergent to a.Then, consider a regrouping.

Now, how am I going to make regrouping precise? Well, that is Consider a subsequence of 1, 2,3,....; I am just considering a subsequence of the natural numbers ; call this subsequence, call it n_k. Define $b_1 := a_1 + a_2 + ... + a_{n_1}$, then $b_2 := a_{n_1+1} + a_{n_1+2} + ... + a_{n_2}$and you get the idea, b_3 is just $a_{n_2+1} + ... + a_{n_3}$.

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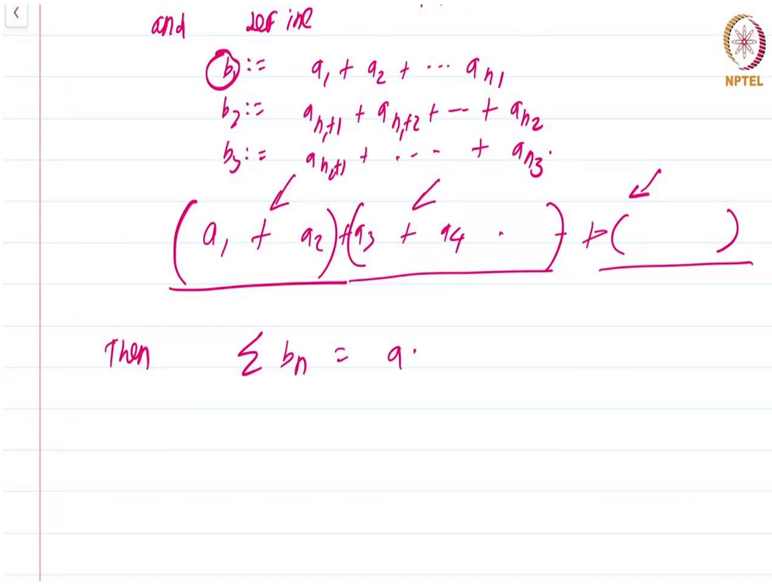

So, essentially what I am doing is; I have this $a_1, a_2, a_3, a_4, \ldots$. A regrouping is just inserting parentheses at appropriate points without changing the order. So, I am just inserting parentheses doing this addition first, then this addition then whatever addition comes next so on and so forth.

So, that is neatly captured by just considering a subsequence of $\mathbb{N}$ and considering a new sequence whose first element is just $a_1, a_2, \ldots, a_{n_1}$, that is just this. Here, the ordering really does not matter, when you are taking this finite sum but I am just writing it in the same order and then b_2 is just this, b_3 is just this so on.

I am just grouping the terms together and there is where this definition perfectly captures what it means for a; for this b_k to be a regrouping. Then, the conclusion $\sum b_k = a$. So, the series $\sum b_k$ and I will not use b_k, let me just write b_n again; the series $\sum b_n$ also converges to a.

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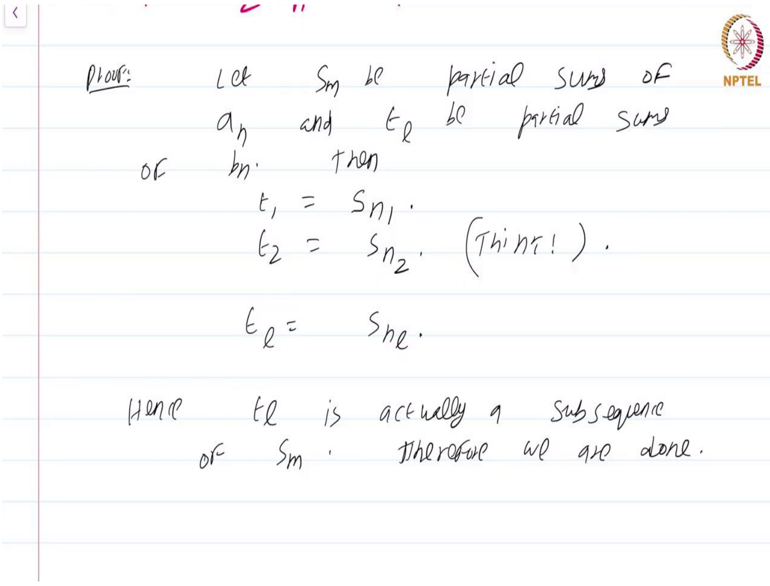

Now, proof is rather easy.

let S_m be partial sums of a_n and t_l be partial sums of b_n. Then, t_1 is actually just S_{n_1} and t_2 is just S_{n_2} think. Again, this is there is nothing really to explain; t_1 is just S_{n_1}, t_2 is just S_{n_2}. In general, this t_l is just S_{n_l} right.

So, hence this t_l is actually a subsequence of S_m. Hence proved therefore, we are done, that is it. Any subsequence of a convergent sequence must converge to the same point. So, this was a really short module; what we have now shown is that you do not really require absolute convergence for grouping terms; just convergence is enough to guarantee good behavior.

This is a course on Real Analysis and you have just watched the module on Grouping Terms of an Infinite Series.