Real Analysis - I
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Lecture – 12.3
The Number E

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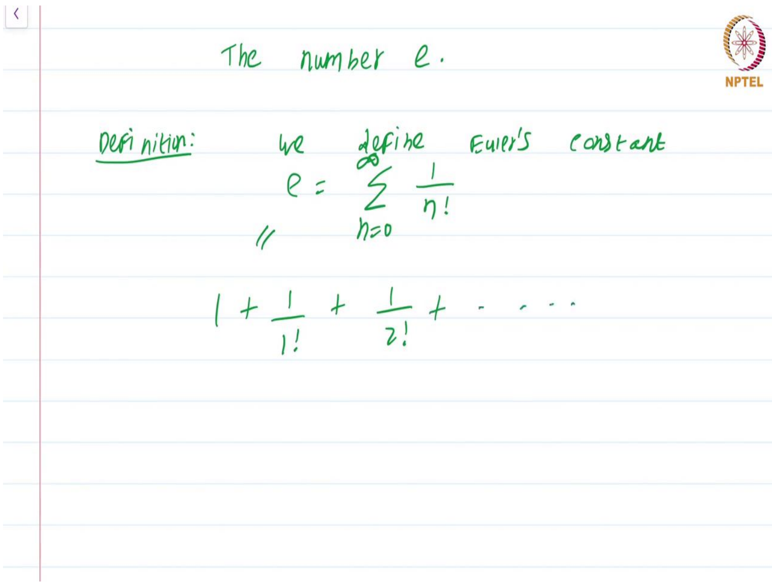

The number e is one of the most fascinating and important constants in all of mathematics. There are several ways to define the number e we shall take the approach of infinite series.

Definition: We define Euler's constant $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Other words it is this familiar $1 + \frac{1}{1!} + \frac{1}{2!} + ...$.

Before we even begin any analysis of this number e, we must first of all show that this infinite series converges. That is rather easy, because once you have written it as $1 + \frac{1}{1!} + \frac{1}{2!} + ...$.

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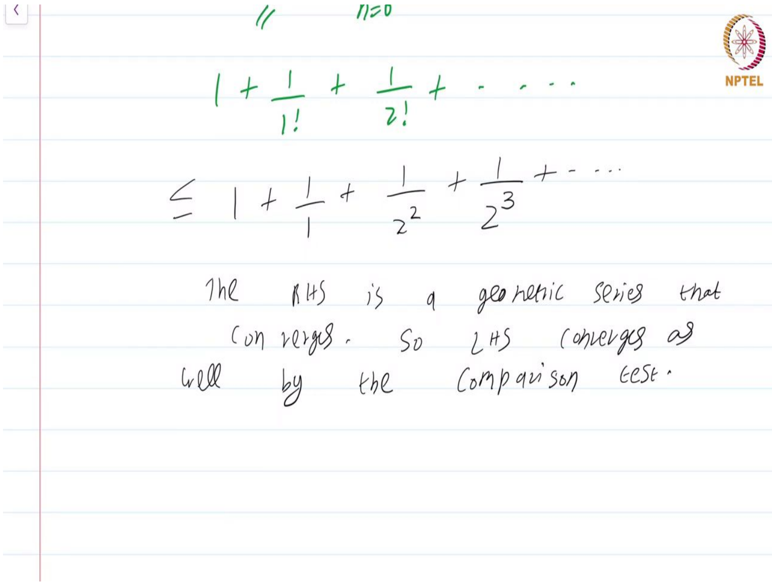

You can just write it as less than or equal to $1+\frac{1}{1}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$. The right hand side is a geometric series that certainly converges. The RHS is a geometric series that converges. So, LHS converges as well by the comparison test.

So, we got a very simple proof that the series $\sum_{n=0}^{\infty}\frac{1}{n!}$ indeed converges and we are just calling , what it converges to be the number e.

Now, this number e is a number of special features that we shall explore once we define the exponential functions and related to this constant e, we shall do that after we develop some amount of calculus. But let me just state and prove one really interesting fact about this number e. It's the fact that this number e is irrational. We have already seen that the square root of 2 is irrational and the proof was not that hard though elegant.

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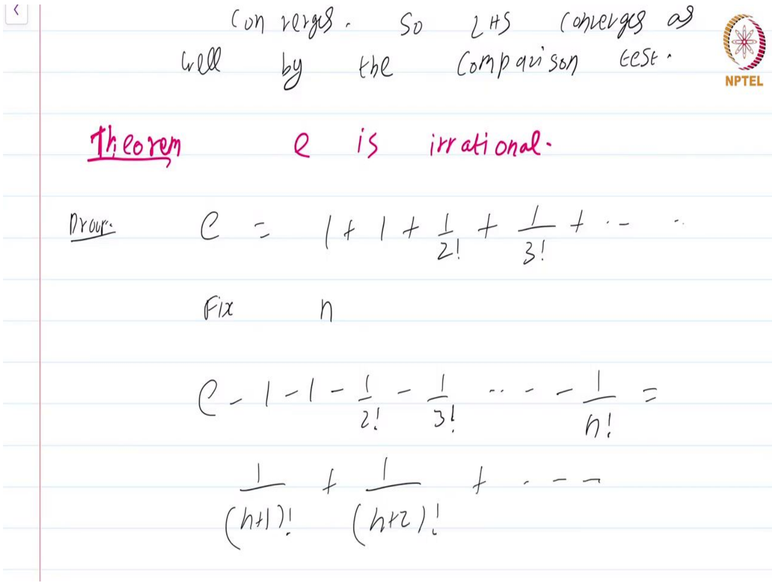

Here we have a very elegant proof, but the somewhat little bit more involved theorem e is irrational,

Let us see a proof. So, what we do is the following: we know that e is just $1+\frac{1}{1!}+\frac{1}{2!}+.....$. Now what we do is the following. We are going to fix n and take the first n terms to the left.

That means we are going to consider $e-1-\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}-.....-\frac{1}{n!}$. We are considering the first n terms in the expansion of e and subtracting it from e. Now this is certainly going to be equal to $\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+.....$. it is going to be this infinite series.

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$$(*)\left(e - 1 - 1 - \frac{1}{2!} - \frac{1}{3!} \cdots - \frac{1}{n!}\right) = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

Clearly $(*)$ is certainly positive.

Assume $e = \frac{p}{q}$, $p, q \in \mathbb{N}$.

Choose $n > q$. multiply $(*)$ by $n!$

So, clearly this quantity * which is e minus this is certainly a positive. * is certainly positive.

Now assume $e = \frac{p}{q},\ p, q \in \mathbb{N}$. I am going to get a contradiction. I am going to assume that e is rational and somehow arrive at a contradiction.

Now, choose we had fixed this n right now choose $n > q$. The same manipulation, obviously holds I am just choosing that fixed n to be a quantity that is greater than q. Of course I can do that.

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Choose $n > q$. multiply $(*)$ by $n!$

$$n!\left(\frac{p}{q} - 1 - 1 - \frac{1}{2!} \cdots - \frac{1}{n!}\right)$$

is a natural number

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$$= \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \cdots$$

Now, what I do is I multiply * by n!. So, I get multiply * by n!. So, I get $n!(\frac{p}{q} - 1 - 1 - \frac{1}{2!} - .... - \frac{1}{n!})$.

Now, since our assumption is that n > q. n! divided by q will be a natural number. Indeed when I expand this out every single term will be a natural number. So, this is a natural number. In fact, a natural number which will be positive of course, natural number itself includes positive, so I need not have said that. It is going to be a natural number.

But this expression $e - 1 - \frac{1}{1!} - \frac{1}{2!} - ......$ was actually equal to $\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ...$. So, let us multiply this by n!, whatever this natural number is it is got to be equal to $\frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + ...$. This is just because the original expression was equal to this $\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ...$.

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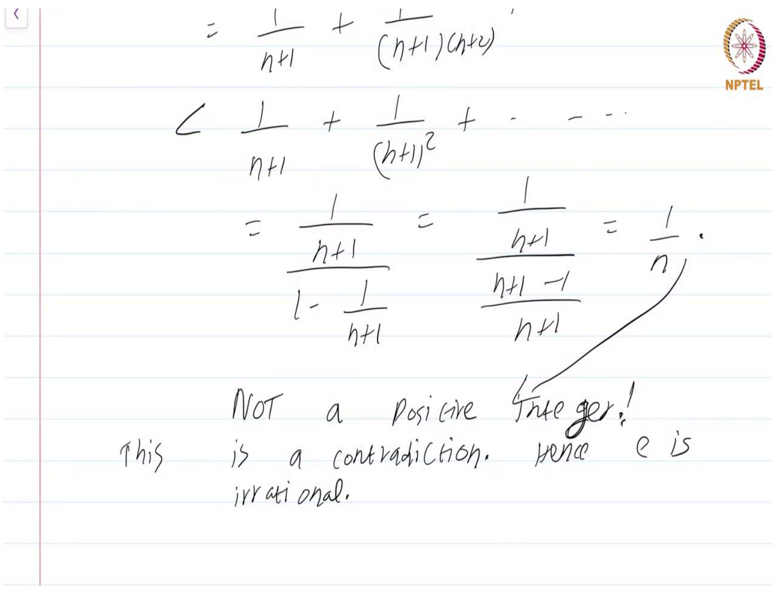

Now, this leads to a problem that is actually equal to $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + ...$ which is certainly going to be less than or equal to. In fact, strictly less than, I need not. I can be more precise less than $\frac{1}{n+1}+\frac{1}{(n+1)^2}+.....$, which is a geometric series. So we have to compute the sum.

This is in fact, equal to $\frac{\frac{1}{n+1}}{1-\frac{1}{n+1}}=\frac{\frac{1}{n+1}}{\frac{n+1-1}{n+1}}$. This is certainly $\frac{1}{n}$ not a positive number, positive integer right, this is a contradiction.

Hence e is irrational. So just some little bit of manipulation with infinite series immediately gave us the fact that the number e is not going to be a rational number , this was a nice proof.

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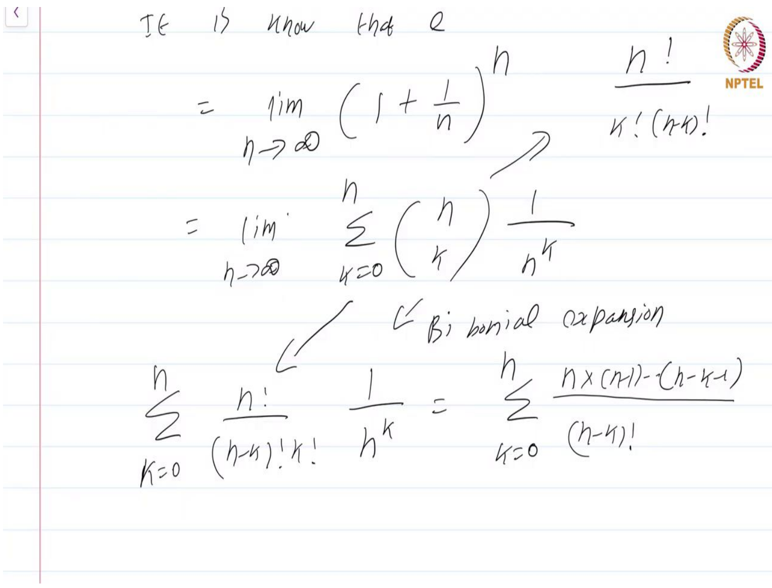

Now let us relate e to another limit that you are familiar with. You are probably familiar with what I must add. It is known that e can also be defined as $\lim_{n\to\infty}(1+\frac{1}{n})^n$.

So, e is also equal to this limit. Why is the series that we have defined also equal to this limit .

Well this is nothing but $\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}$. This is just binomial expansion. I am just using the familiar binomial expansion, I am writing out what $(1+\frac{1}{n})^n$ is .

Now, this is interesting. We got some summation, but it does not look very similar to what we have. But given that there is $\binom{n}{k}$ which involves factorials there is some hope. So, let us expand this inner part further. So, this is $\sum \frac{n!}{(n-k)!k!}\frac{1}{n^k}$. So, which is just $\sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k-1)}{k!n^k}$.

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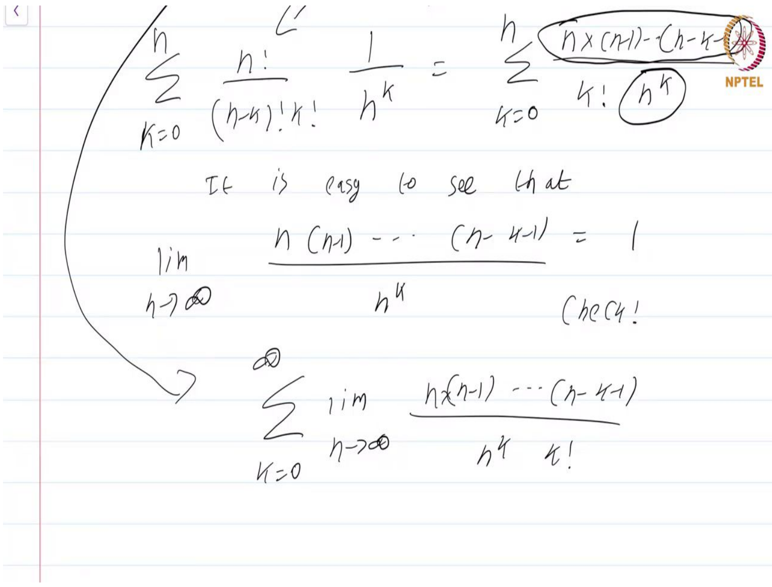

Now, observe that there are k terms here; $n(n-1)\cdots(n-k-1)$. All the way and in the denominator we have n^k. So, it is easy to see that $\lim_{n\to\infty} \frac{n(n-1)\cdots(n-k-1)}{n^k} = 1$. I leave it to you to check this is a fairly easy exercise.

So, what we have is we have summation k = 0 to n, some quantity divided by k! and that quantity goes to 1 as n goes to infinity. Well imagine you have this expression limit n going to infinity summation k = 0 to n.

What if I could pull this trick, what if I could write this as $\sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k-1)}{n^k k!}$. We know that this limit has got to be equal to k!. We will end up with $\sum_{k=0}^{\infty} \frac{1}{k!}$ which is exactly e.

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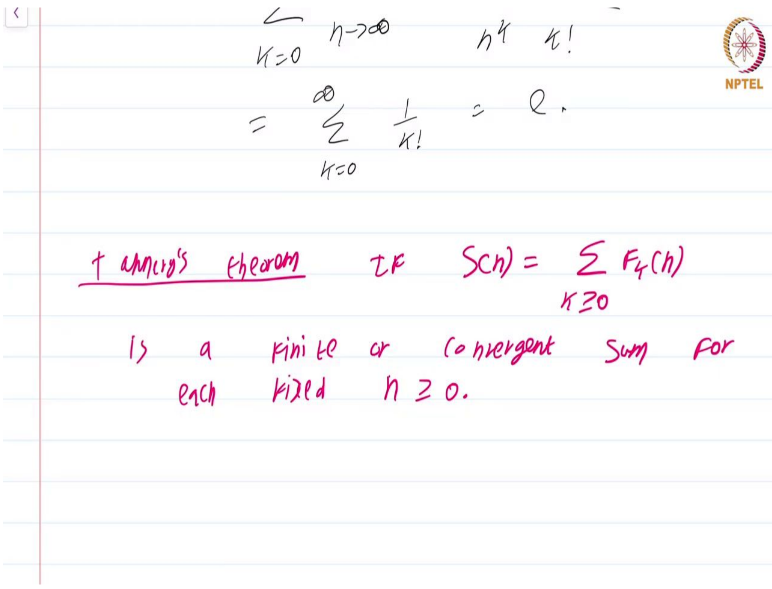

The only problem is this step where I am just sneakily taking the limit inside and applying the limit requires a proof. It is not always true that you can just interchange stuff like this please recall one of the examples given in the very first week of lectures where I showed that carelessly interchanging summations can lead to nonsensical results. So, we need to justify why this interchange of limit and summation is a valid operation.

In fact, the whole of analysis is to make sure that such operations are valid and to provide the reasoning why they are valid and the hypothesis under which they are valid. So, this justification is provided by a theorem called Tannery's theorem, the theorem runs as follows.

If $S(n) = \sum_{k\geq 0} F_k(n)$ is a finite or convergent sum for each fixed natural number or fixed $n \geq 0$.

So, what we have is for each n, I am given a series expression, this is either a finite or a convergence, I don't really care which it is. So, for each n I am given a sum that runs over k, this sum could be finite, this sum could be convergent it really does not matter.

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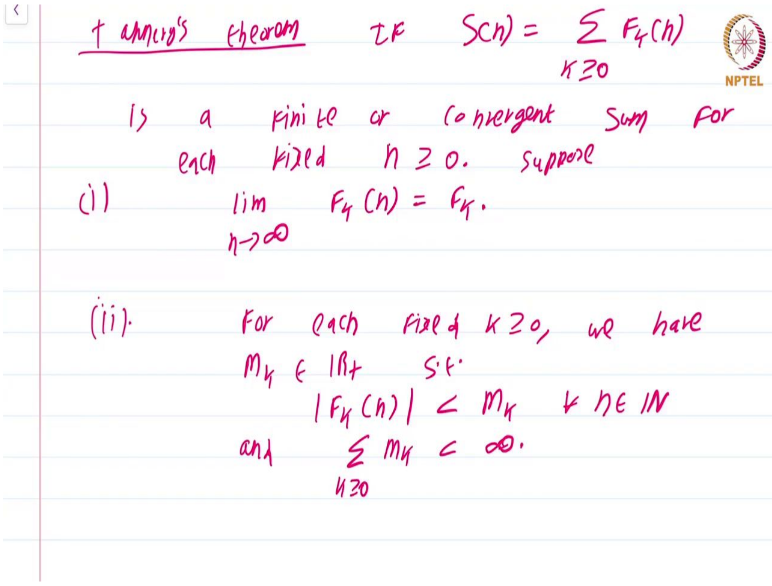

Suppose (i) $\lim_{n \to \infty} F_k(n) = F_k$.

(ii) For each fixed $k \geq 0$ we have $M_k \in \mathbb{R}_+$ such that $|F_k(n)| < M_k \ \forall n \in \mathbb{N}$ and $\sum_{k \geq 0} M_k < \infty$.

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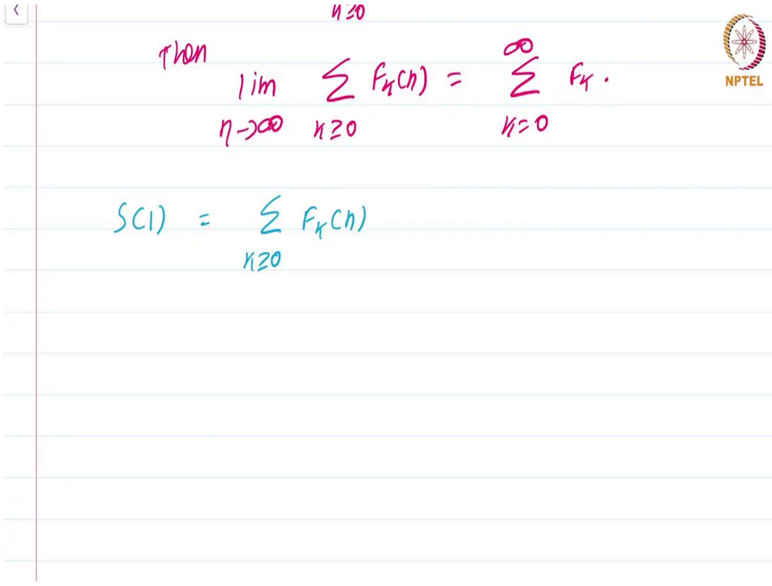

Then the conclusion is then the $\lim_{n\to\infty}\sum_{k\geq 0}F_k(n)=\sum_{k=0}^{\infty}F_k$. Essentially we have taken the limit inside. To understand what Tannery's theorem is trying to say it might be a good idea to draw a grid. So, we have this $S(1)$ which is supposed to be $\sum_{k\geq 0}F_k(n)$. So, essentially let us expand this sum.

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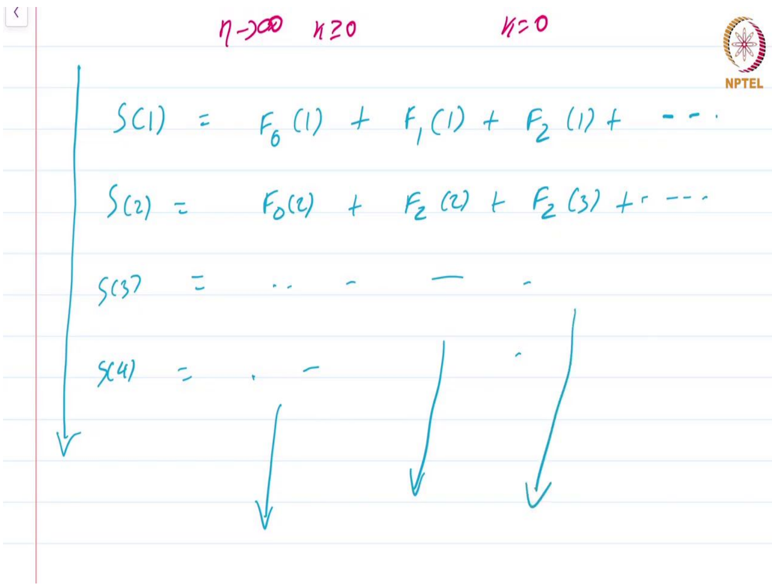

So, S(1) will nothing but $F_0(1)+F_1(1)+F_2(1)+\cdots$. Similarly S(2) would be just $F_0(2)+F_2(2)+F_2(3)+\cdots$ and similarly you have S(3), S(4) and so on you have these infinite sums.

Now, what we are trying to do is we are trying to determine the limit as you go down , that is what $\lim_{n\to\infty}\sum_{k\geq 0}F_k(n)$ is supposed to be this is just S(n) . We are trying to find the limit as you go down the grid.

We are also given that if you go down each one of these columns in this grid, you have a limit we are calling this limit F_0 , we are calling this limit F_1 we are calling this limit F_2 and so on.

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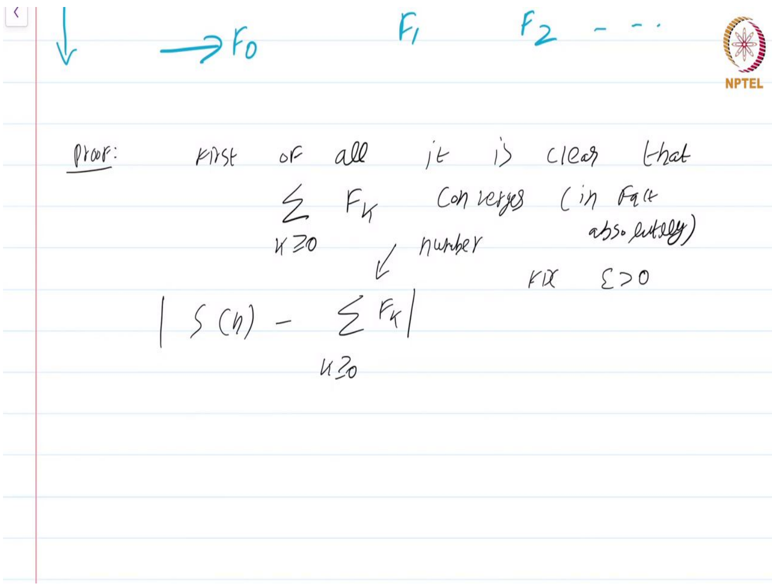

What Tannery's theorem says is that under this specific condition (ii) which just says that for each fixed k we have a bound on $F_k(n)$, M_k such that $\sum_{k\geq 0}M_k$ actually converges. Then to determine this limit of the essence it is enough to just sum up these. So, essentially we are giving a condition under which you can interchange the limit and the summation.Let us see a proof.

First of all it is clear that $\sum_{k\geq 0} F_k$ converges. In fact, absolutely in fact, why is this the case well look at hypothesis (ii). What it is saying is that for each fixed k, $F_k(n)$ is actually bounded in absolute value by M_k and $\sum_{k\geq 0} M_k < \infty$.

In particular this F_k which is nothing but $\lim_{n\to\infty} F_k(n)$ must also be bounded in absolute value by M_k and since $\sum_{k\geq 0} M_k$ is convergent by comparison test. $\sum F_k$ will also be convergent. So, I am going to leave these minor checking's to you. So, first of all we have that $\sum_{k\geq 0} F_k$ converges. Now the term we are interested in is $S(n) - \sum_{k\geq 0} F_k$.

What we want to show is that this is actually a number even though I have written it as a series, what it stands for is actually the term that it converges to. We want to show that given any ϵ, so fix $\epsilon > 0$ we have to find an n such that $|S(n) - \sum_{k\geq 0} F_k| < \epsilon$.

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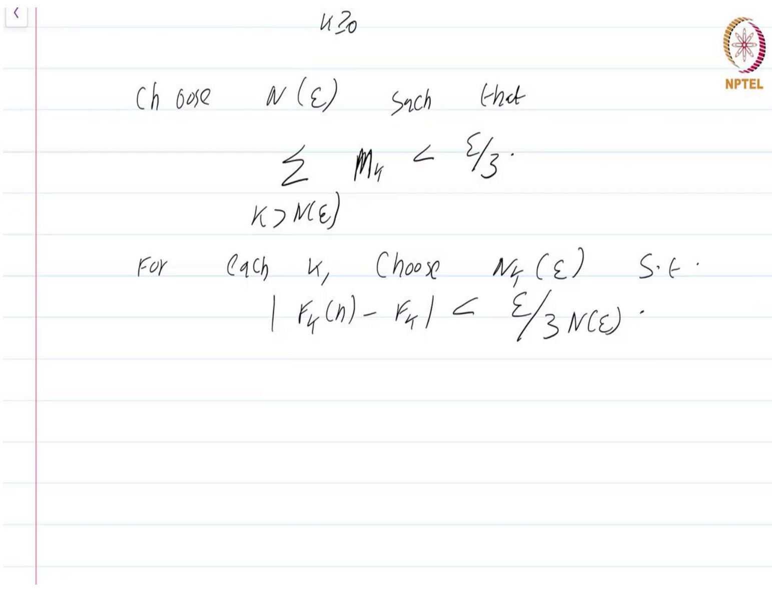

So, to do this first choose $N(\epsilon)$ such that such that $\sum_{k>N(\epsilon)} M_k < \frac{\epsilon}{3}$. This can certainly be done because these $\sum M_k$'s converge. So, you can make the tail as small as you want . Now, what you do is for each k choose $N_k(\epsilon)$, such that $|F_k(n) - F_k| < \frac{\epsilon}{3N(\epsilon)}$.

Now you might be wondering why I am pulling constants out of the air, you will understand in a moment; why I am pulling out this I mean I am putting ϵ by $3N(\epsilon)$. So, what we have is that each $F_k(n)$ converges to F_k .

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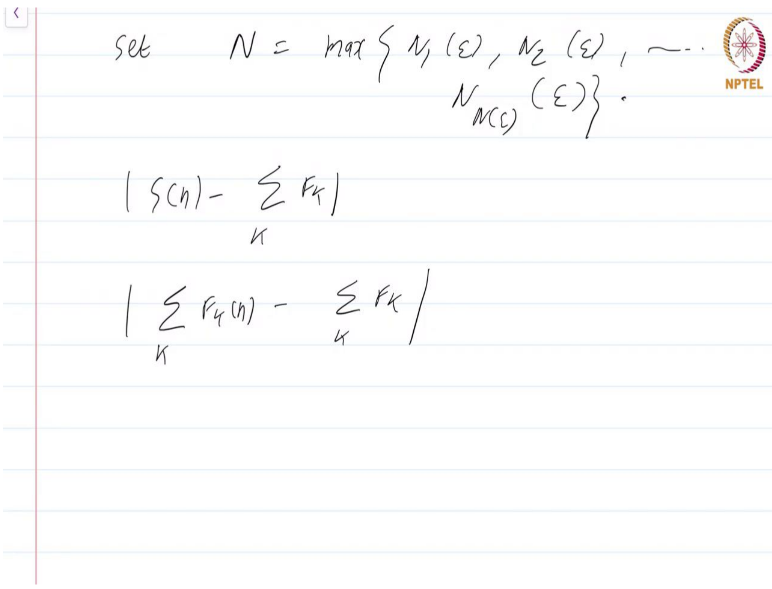

Now what we do is set $N := max\{N_1(\epsilon), N_2(\epsilon), ..., N_{N(\epsilon)}(\epsilon)\}$. Recall this $N(\epsilon)$ was the point at which the tail of the series $\sum M_k$ can be made less than $\frac{\epsilon}{3}$ For each k we can choose $N_k(\epsilon)$ such that $F_k(n)$ and F_k are at max $\frac{\epsilon}{3}$ and ϵ . Now you are choosing N to be the maximum of the quantities $N_1(\epsilon), N_2(\epsilon), ..., N_{N(\epsilon)}(\epsilon)$.

Now, what does all this complicated jugglery give us we wanted $|S(n) - \sum_k F_k|$. But $S(n)$ is actually just $\sum_k F_k(n)$ and $-\sum_k F_k$, fine.

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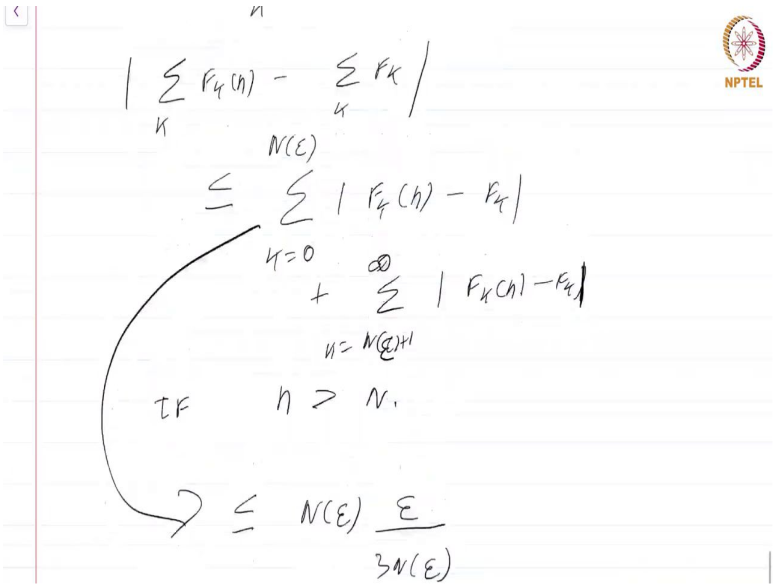

Now this is certainly going to be less than or equal to $\sum_{k=0}^{N(\epsilon)} |F_k(n) - F_k| + \sum_{N(\epsilon)+1}^{\infty} |F_k(n) - F_k|$.

Now, how does this help well if n > N then something interesting happens, how was N chosen. It was the maximum of $N_1(\epsilon), N_2(\epsilon), \ldots, N_{N(\epsilon)}(\epsilon)$. If n > N, then what we have is $F_k(n) - F_k$ must be less than $\frac{\epsilon}{3N(\epsilon)}$.

So, what do we get? What we get is this first quantity will certainly be less than or equal to $N(\epsilon)\frac{\epsilon}{3N(\epsilon)}$. Why is this the case well because, each term $|F_k(n) - F_k|$ will indeed be less than $\frac{\epsilon}{3N(\epsilon)}$ and there are $N(\epsilon)$ such terms. So, this will be less than or equal to $\frac{\epsilon}{3N(\epsilon)}$.

Now, what about this second term where you are summing up from $N(\epsilon)+1$ to infinity, recall that $N(\epsilon)$ was chosen to make the tail of the series $\sum M_k$ really small right. We had chosen it to make the summation, I mean the series really small the tail of that series and here if you observe you have $F_k(n) - F_k$ and we know that the absolute values of $|F_k(n)| < M_k$ and the $|F_k| \leq M_k$.

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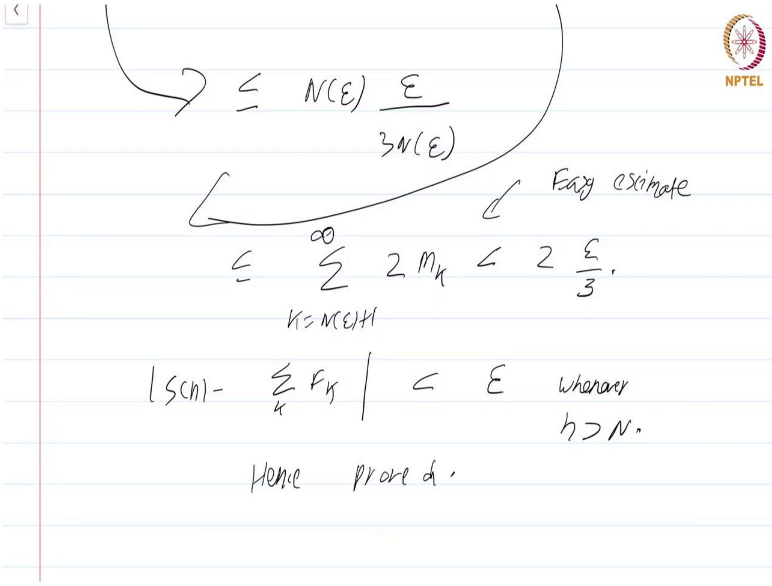

So, we can write this term as summation less than or equal to $\sum_{k=N(\epsilon)+1}^{\infty} 2M_k$, this is just an easy estimate. This just follows from the fact that $|a-b| \leq |a| + |b|$.

But this is less than $2\frac{\epsilon}{3}$. Putting all this together we get that $|S(n) - \sum_k F_k| < \epsilon$ whenever n > N, hence proved . So, the proof is not difficult, it just involves a bit of jugglery because we gave a lot of data.

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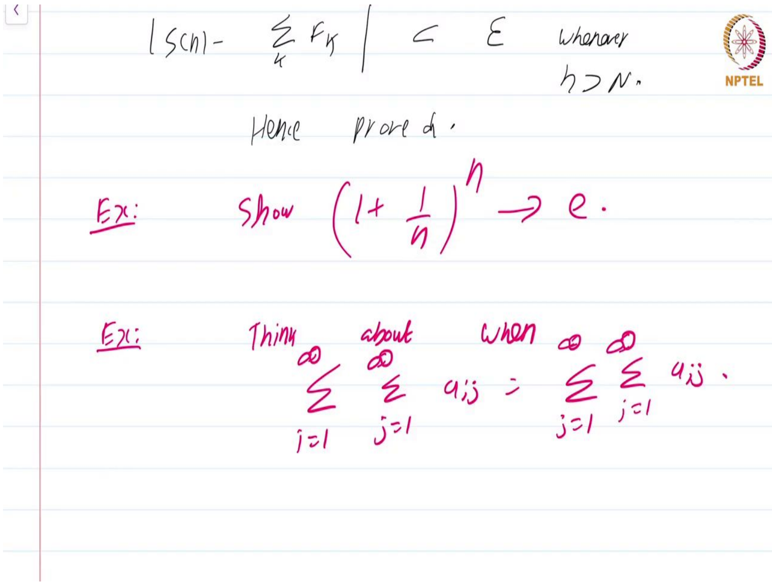

Exercise: Show $(1+\frac{1}{n})^n \to e$.

This is actually most of the work has been done and you just have to apply Tannery's theorem and you will get it. I am going to leave you with another exercise which will be expanded upon in the assignment, you are going to prove you are going to prove a condition or let me just leave it as an open ended exercise now.

So that you can think about it later and think about when $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty}$.

So, we have this double series sort of: we are first summing up over j then summing up over i, I am asking when can you interchange think about when you will be able to interchange think about it using Tannery's theorem. I shall give you an exercise where you give a precise condition under which this is valid and you will prove it using Tannery's theorem.

This is a course on real analysis and you have just watched the module on the number e.