Real Analysis - I
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Week - 04
Lecture – 12.1
Absolute and Conditional Convergence

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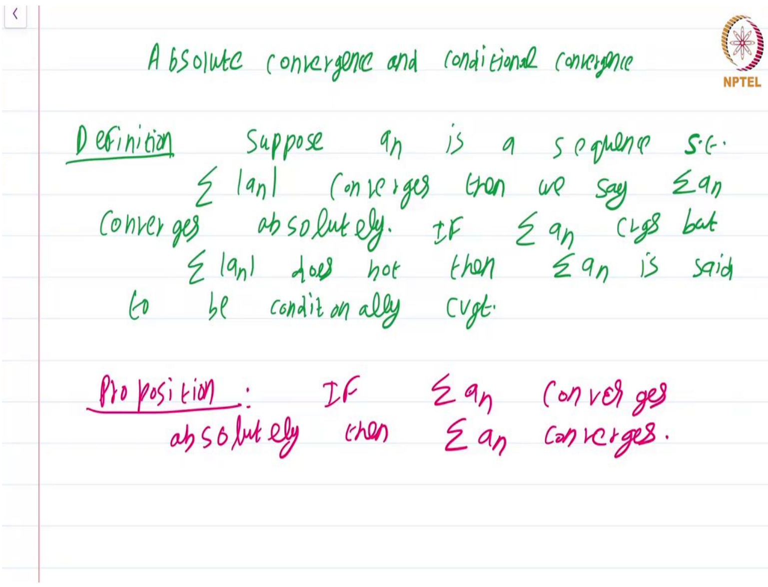

In the last module, we ended with the definition and a proposition regarding absolute and conditional convergence. Let me just recall the definition, because it's going to be the central aspect of this particular lecture.

Definition: Suppose a_n is a sequence such that $\sum |a_n|$ converges, then we say $\sum a_n$ converges absolutely. Not a very creative name, but nevertheless it serves the purpose.

If $\sum a_n$ converges, but $\sum |a_n|$ does not then $\sum a_n$ is said to be conditionally convergent. We also stated a proposition.

Proposition: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Let us see a proof of this. It is actually done already.

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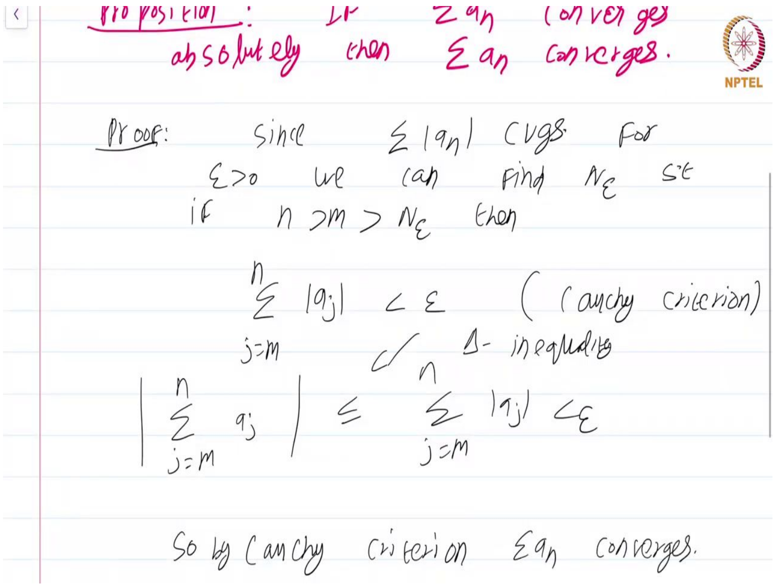

But since this is the central portion of this, I will give a quick proof, leaving the details to you . Since, $\sum |a_n|$ converges, that's the hypothesis for $\epsilon > 0$, we can find N_ϵ such that, if $n > m > N_\epsilon$ then $\sum_{j=m}^{n} |a_j| < \epsilon$, this is just the Cauchy criterion, one side of the Cauchy criterion. Remember Cauchy criterion is an if and only if condition.

But, $|\sum_{j=m}^{n} a_j| \leq \sum_{j=m}^{n} |a_j| < \epsilon$, this is just triangle inequality. So, again by the other direction of Cauchy criterion, so by Cauchy criterion, $\sum |a_n|$ converges.

So, this was a fairly straightforward and easy proof. Now, we are going to be studying series now that converge, but conditionally not absolutely. And the study of such series is really interesting, that is best seen by studying an actual example.

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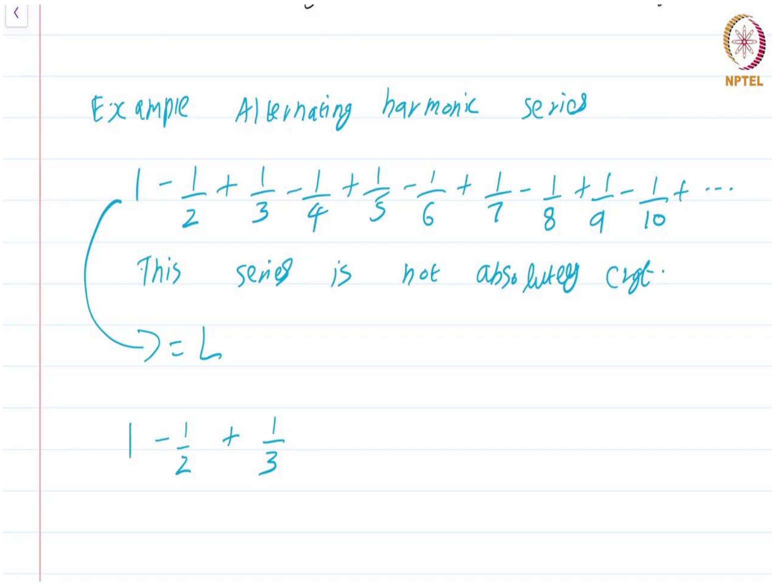

Let us take the example of the alternating harmonic series. This is the sum $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\ldots\ldots$ This is the alternating harmonic series. It is just the series $\sum\frac{1}{n}$ where I have inserted minuses at every other place.

Now, we know that this series is not absolutely convergent, because the harmonic series diverges. This series is not absolutely convergent. Now, let us for the time being assume that this converges to some L , let us say.

I am not going to compute L now nor am I going to show that this series in fact, converges to something at all right now. We will see it in a more general theorem because proving that this series converges is the same as proving a much more general fact, there is no additional difficulty. I will do that in a moment that is the alternating series test, but let us just assume for the time being that this converges to L bear with me for a few minutes.

Now, what I am going to do is, I am going to explore a question that I had raised all the way back in the beginning about whether you can manipulate infinite series with the same level of freedom, which you can manipulate finite series. You can rearrange the terms of a finite series, you can group terms together and so on all the familiar laws of arithmetic hold.

But, what about infinite series manipulation? Let us see what happens if we manipulate this series. What I am going to do is the following.

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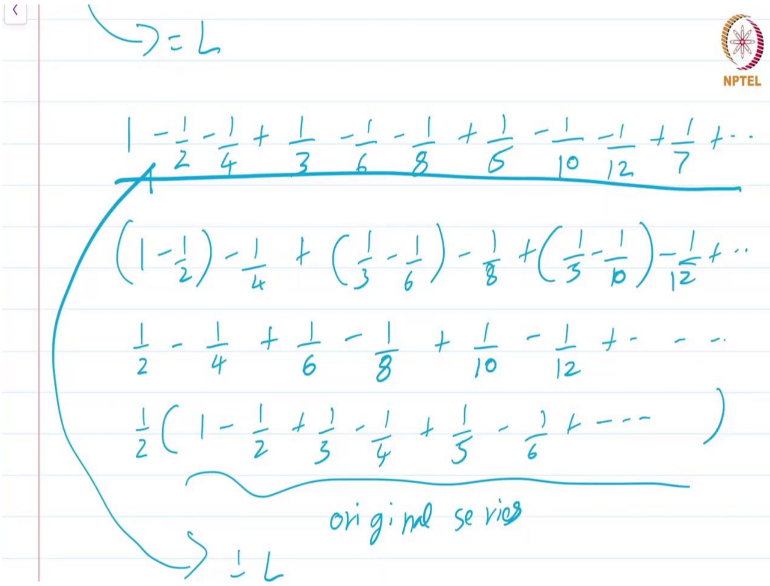

What I will do is, I will move this $-\frac{1}{4}$ here, then put $\frac{1}{3}$. So, all I have done is I have grouped together two negative terms. Then what I am going to do is, I am going to group the next two negative terms $-\frac{1}{6}$ and $-\frac{1}{8}$, then the positive term is going to come. Then I am going to group the next two negative terms which I am not mistaken is $-\frac{1}{10}$ and $-\frac{1}{12}$ and the next positive term is $\frac{1}{7}$.

Note very carefully that every term in the original series will occur here, but in a different position, because instead of alternating one positive and one negative I am taking one positive two negative, one positive two negative, one positive two negative so on.

So, what I will do now is I am going to group these terms together after doing this rearrangement. I am going to write it as $(1-\frac{1}{2})-\frac{1}{4}+(\frac{1}{3}-\frac{1}{6})-\frac{1}{8}+(\frac{1}{5}-\frac{1}{10})-\frac{1}{12}+.....$. And you get the picture of what is happening.

Now, this is just $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+.....$. You can just see that this is what will happen if you do this manipulation. Please check this rigorously, I am being a bit hand wavy because there is nothing really deep happening. This is just basic arithmetic, wonderful.

Now, this is just $\frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+.....)$. Low and behold this is the original series. So, if this original series were to converge to L, then this rearrangement where I have just bunched together terms in a different way, this should converge to half of L. So, something interesting is happening. If you rearrange the terms of this series, which is not absolutely convergent, but it is actually convergent which we are going to see now.

Then if you rearrange, you get a different value for the sum in fact you get half of the original value. Now, I leave it to you to check what will happen if I group these terms differently, instead of one positive and two negative, what if you do one positive three negative, two positive five negative.

If you are not able to make headway, please Google search, rearranging the alternating series, I will give you a link below. And just see the various results that have been done, what and all can you get by rearranging this.

In the next module, I will prove a very abstract theorem or rather sketch the proof of a very abstract theorem saying that you can rearrange and get anything that is a precise statement called the Riemann's rearrangement theorem. You can get anything by just rearranging the terms here.

So, but I am talking about specific rearrangements in this module, where some positive terms are grouped together, then negative terms are grouped together, that is it is a fixed pattern, and you will get some interesting stuff.

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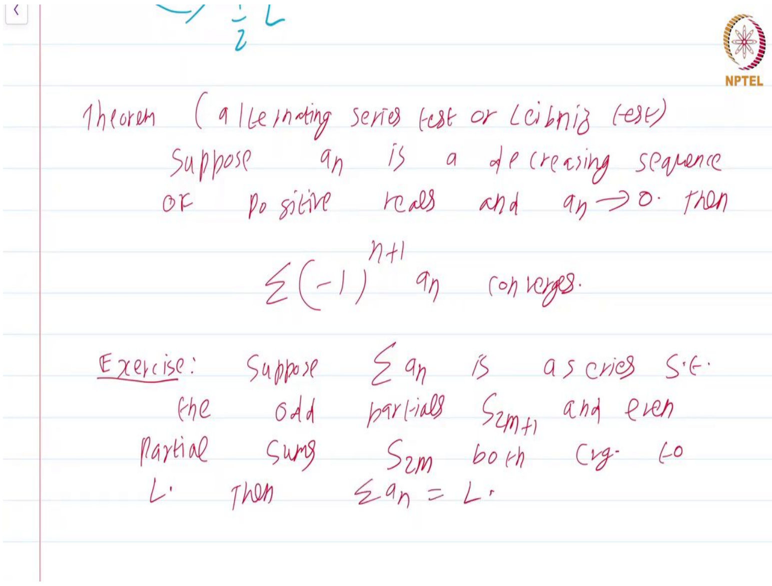

So, now it is interesting to see whether this alternating series actually converges this alternating harmonic series, that is the content of the next theorem, which is also called the alternating series test or Leibniz test. This is as follows.

Suppose, a_n is a decreasing sequence of positive reals and $a_n \to 0$, then $\sum(-1)^{n+1}a_n$ converges.

If you have a series of alternating terms such that, after you throw away the sign, if you consider their sequence a_n goes to 0 ,moreover it is all going to be greater than or equal to 0 and decreasing, then the series $\sum(-1)^{n+1}a_n$ converges.

Before I give the proof, I want you to solve this exercise. Solve it right now in your head you need not write down a proof instantly, but just solve it right now in the end, right now in your head.

Suppose, a_n is a sequence and the so or rather suppose let me just change it slightly. Suppose, $\sum a_n$ is a series such that the odd partial sums S_m or rather S_{2m}, I am just taking the partials sums of even number of terms collected together, and even partial sums S_{2m} both converge to L, then $\sum a_n = L$.

Suppose, I am giving you a series such that I know that if I collect together even number of terms, that means, the first sum of the first two terms; the sum of the first 4 terms, the sum of the first 6, 8, 10, 12 so on that sequence converges to L, then I consider the first three, the first five, the first seven so on. That is the odd partial sums if that also converges to L, then the whole series converges to L this is a rather straightforward exercise that falls out on your lap immediately from the very definition of convergence.

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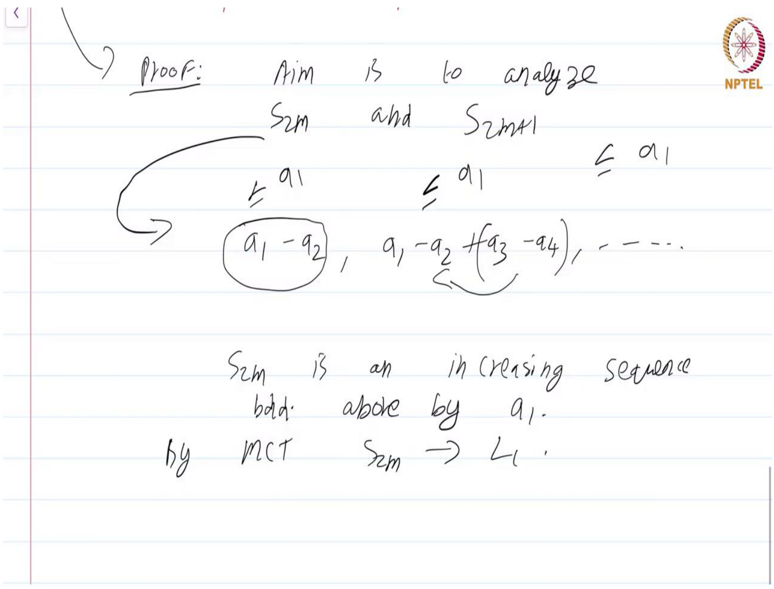

Now, let us prove the theorem. Let us give the proof of the theorem. I am going to assume this exercise is why I want you to solve this exercise immediately in your head at least if not on pencil and paper.

Now, the aim is to analyze S_{2m} and S_{2m+1}. Let us just analyze S_{2m} first. So, what is the S_{2m} going to be? It is going to be first term is $a_1 - a_2$, then $a_1 - a_2 + a_3 - a_4$ and so on. Now, observe the following this $a_1 - a_2$ is certainly going to be less than or equal to a_1, right.

Now, when you are adding back a_3, $|a_3| < a_2$ right, so I have subtracted a_2, but added back something of value less than a_2, then I am subtracting a_4 again, whatever happens this is also going to be less than or equal to a_1.

And if you observe carefully at every point of time when I am adding two terms, the magnitude of those two terms will always be a positive quantity, because for instance here I am adding $a_3 - a_4$. I am always going to be adding a positive quantity, but that positive quantity cannot offset the negativity coming from the previous quantity.

So, at each stage, I will always get less than or equal to a_1. Moreover it is an increasing sequence both should be very patently clear from what I have said. So S_{2m} is an increasing sequence bounded above by a_1. So, by the monotone convergence theorem, S_{2m} converges to some L_1.

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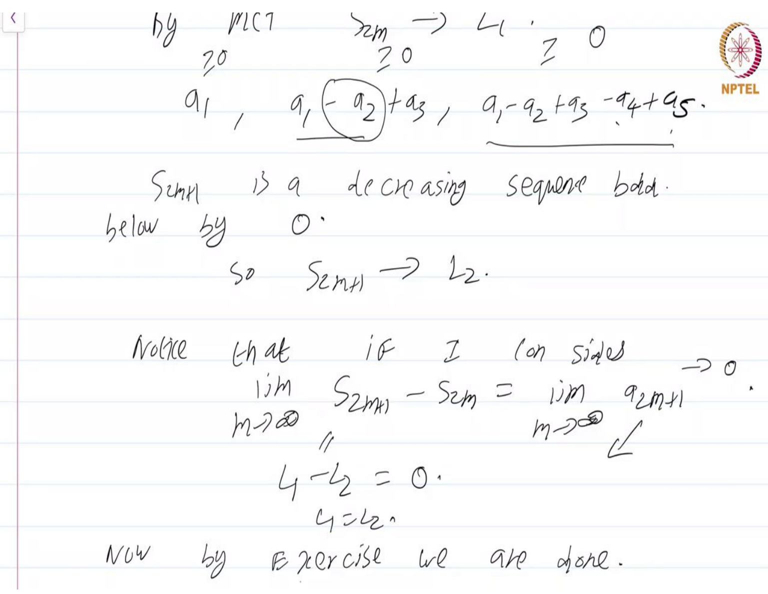

Now, let us look at S_{2m+1}. So, the first term will be a_1, then it will be $a_1 - a_2 + a_3$, then $a_1 - a_2 + a_3 - a_4 + a_5$. So, these are the first few terms. Now, observe that these are all going to be greater than or equal to 0. Let us see why that is the case.

Let us see why that is the case. Now, $a_1 \geq 0$ simply because a_1 is a positive quantity. Now, I am subtracting a quantity a_2 whose magnitude is less than a_1.

So, I am left with a positive quantity, then I am adding back another positive quantity a_3. So, this will be greater than or equal to 0. In a similar way, you can see that all these terms will be greater than or equal to 0. I will always be subtracting a quantity and then adding back. So, in any case, it will always be a positive quantity.

Moreover, look at what is happening. I have subtracted a_2 from a_1, and added back a quantity, whose magnitude is less than a_2 because $a_3 \leq a_2$. Similarly, in this I have subtracted a_4, but added back a quantity whose magnitude is lesser. So, S_{2m+1} is a decreasing sequence bounded below by 0. So, S_{2m+1} converges to some L_2.

Now, notice that if I consider $\lim_{m\to\infty} S_{2m+1} - S_{2m}$, I am left with $\lim_{m\to\infty} a_{2m+1} = 0$. Why does a_{2m+1} go to 0? Because that is in their hypothesis that the sequence a_m converges to 0.

So, a_{2m+1} will also converge to 0. What does this tell us? This tells us that $\lim_{m\to\infty} S_{2m} - S_{2m+1}$ which is nothing but $L_1 - L_2$. This is 0, or in other words $L_1 = L_2$. Now, by exercise, we are done.

So, we have now shown that if you have an alternating series with the terms of the sequence going to 0 and decreasing and positive, then the alternating series converges. So, this concludes this module. In the next module, we will see the famous Riemann's rearrangement theorem, and also see that if you have an absolutely convergent series then you can rearrange without this fear.

This is a course on Real Analysis. And you have just watched the module on Absolute Convergence and Conditional Convergence.