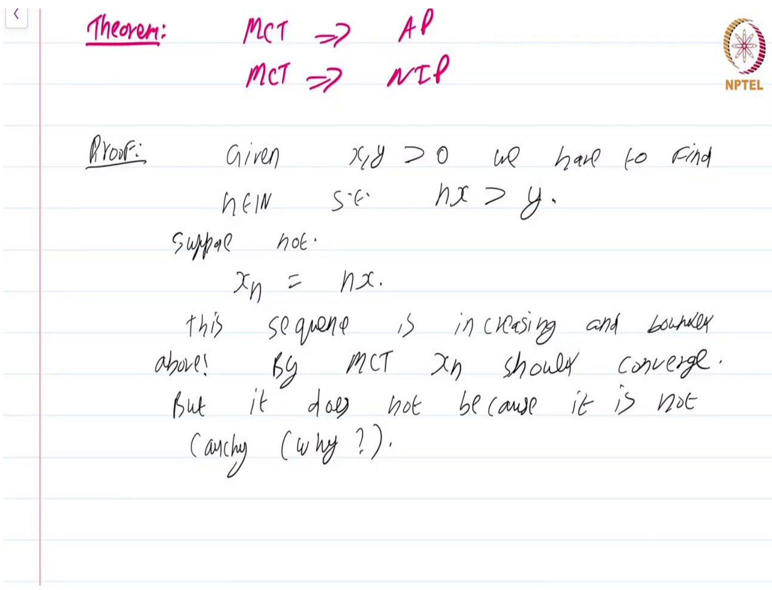
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 10.4
MCT Implies Completeness

(Refer Slide Time: 00:13)



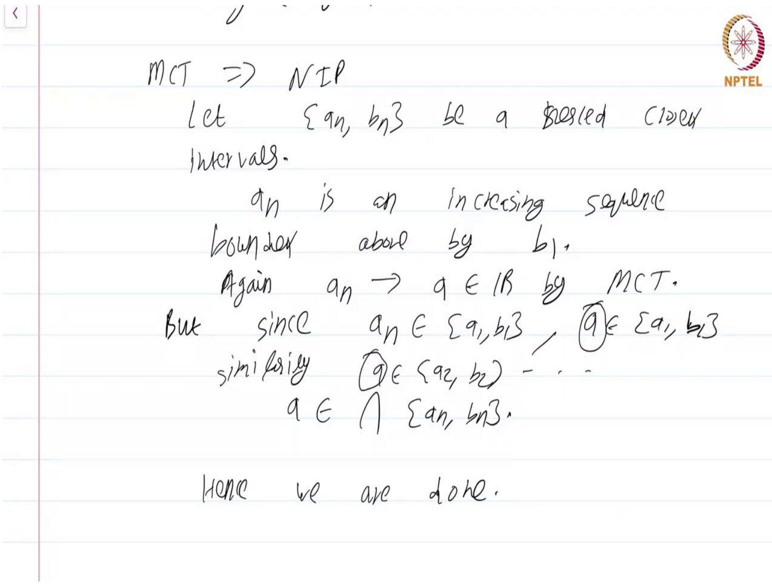
Let us see yet another version of completeness. We are going to show that the monotone convergence theorem actually implies completeness. So, we could have taken the monotone convergence theorem as an axiom, instead of the axiom of completeness. Instead of directly showing that, the monotone convergence theorem implies completeness which I am going to cleverly leave to you as an exercise. I will show a theorem that MCT implies AP and MCT implies Nested Intervals Property ok.

This will together show that the monotone convergence theorem implies completeness by the fact that AP + NIP implies completeness . Now, let us first show that monotone convergence theorem implies the Archimedean property. So, what do we have to show?

Given x , y > 0, we have to find $n \in \mathbb{N}$ such that nx > y . Now, suppose not. Consider the sequence $x_n = nx$. This sequence is increasing and bounded above. Why? Because y is an upper bound.

By monotone convergence theorem (x_n) should converge, but it does not because it is not Cauchy. I leave you to check why it is not Cauchy; it is rather easy . That was really short; the monotone convergence theorem immediately gives the Archimedean property.

(Refer Slide Time: 02:29)



Now, next equally easy is the fact that monotone convergence theorem implies the nested intervals property.

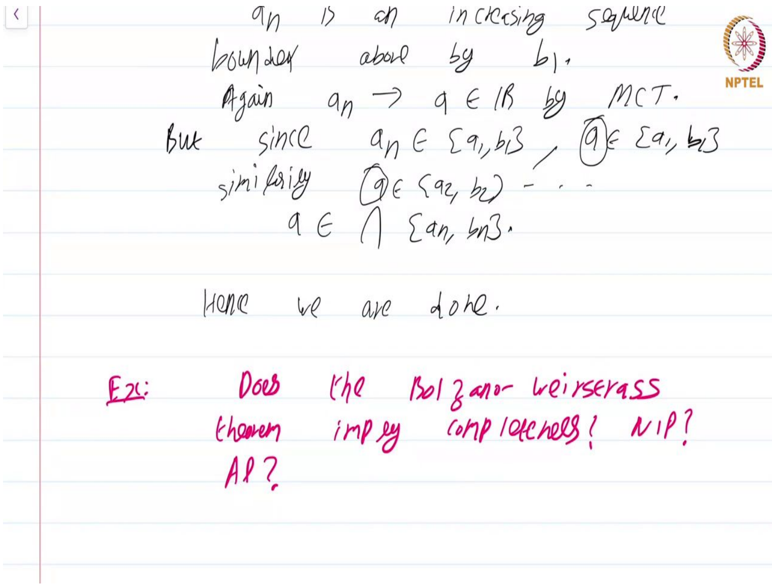
So, let $[a_n, b_n]$ be a sequence or be nested closed intervals. Here, again it is very clear; (a_n) is an increasing sequence, bounded above by b_1. Let us say any of the b_n will actually act as an upperbound. I am taking it to be b_1. Again (a_n) converges to some $a \in \mathbb{R}$ by monotone convergence theorem.

But, since a_n is entirely contained in $[a_1, b_1]$. That means, a is contained in $[a_1, b_1]$; similarly, a is contained in $[a_2, b_2]$ and so on. So, a is in fact there in the $\cap[a_n, b_n]$. So, I want you to think why a is there in $[a_1, b_1]$ and similarly, why a is there in $[a_2, b_2]$. You can either use the fact that these intervals are closed or use the fact that each one of these a_n's have every single b, b_i as an upper bound.

You can use multiple arguments to see that this element a will have to belong to every single $[a_n, b_n]$. Hence, we are done. So, the monotone convergence theorem actually implies completeness. Many people do take the monotone convergence theorem as the axiom of completeness, instead of our usual axiom of completeness.

So, it is very good to see multiple viewpoints of the same central concept to have a good understanding from multiple angles. Let me leave you in light of this remark with an exercise.

(Refer Slide Time: 05:05)



Does the Bolzano Weierstrass theorem, that is the fact that any bounded sequence has a convergent subsequence; does the Bolzano Weierstrass theorem imply completeness? Of course, the same can be asked: does it imply the nested intervals property; does it imply the Archimedean property? Please do think about this.

This is a course on Real Analysis and you have just watched some module on MCT Implies Completeness.